Supersymmetric Quantum Theory, 
Non-Commutative Geometry, 
and 
Gravitation 

Lecture Notes 
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Contents

Preface

1. The classical theory of gravitation

2. (Non-relativistic) quantum theory

3. Reconciling quantum theory with general relativity: quantum space-time-matter

4. Classical differential topology and -geometry and supersymmetric quantum theory
   4.1 Pauli’s electron
   4.2 The special case where \( M \) is a Lie group
   4.3 Supersymmetric quantum theory and geometry put into perspective

5. Supersymmetry and non-commutative geometry
   5.1 Spin\(^c\) non-commutative geometry
   5.2 Non-commutative Riemannian geometry
   5.3 Reparametrization invariance, BRST-cohomology, and target-space supersymmetry

6. The non-commutative torus
   6.1 Spin geometry (\( N = 1 \))
   6.2 Riemannian geometry (\( N = (1, 1) \))
   6.3 Kähler geometry (\( N = (2, 2) \))

7. Applications of non-commutative geometry to quantum theories of gravitation
   7.1 From point-particles to strings
   7.2 A Schwinger-Dyson equation for string Green functions from reparametrization invariance and world sheet supersymmetry
   7.3 Some remarks on \( M \)(atrix) models
   7.4 Two-dimensional conformal field theories
   7.5 Reconstruction of (non-commutative) target spaces from conformal field theory
   7.6 Superconformal field theory, and the topology of target spaces

8. Conclusions
Preface

These are notes to a course taught by the under-signed at a Les Houches summer school organized by A. Connes and K. Gawędżki, in 1995. They follow the program of the lectures presented at Les Houches and of the notes written there, but they are considerably more detailed than the lectures and those notes. In working out the details, I received very valuable help from my two co-authors. Our work led us to finding some new results which will, in part, be published as research papers and, in part, are described in these notes. Thus, these notes contain a mixture of review of very well-known and less well-known matters and of recent or new results by many authors (including ourselves).

The writing of these notes was not exactly a leisurely hike. It more resembled an excursion to the top of Mont Blanc; (I hasten to confess that I have actually never made it to the top of Mont Blanc, in reality; but I have some idea of how that would feel from other experiences in the mountains): One starts to climb the foothills following very well-known (and, perhaps, not entirely well-known) tracks — Sections 1–3 — until one reaches a refuge – Section 4 – where one takes a rest. The following day, one starts to continue the ascent, very early in the day, through more difficult terrain; the air is getting thinner, and one discovers unfamiliar tracks — Sections 5 and 6. Finally, one approaches the top, along one of the standard routes, feeling somewhat exhausted — Section 7, Sects. 7.1 and 7.2. Having reached the top, one is a little out of breath and decides to admire the view — an unwritten section. After a good while, and with some new energy, one starts the descent, choosing a new, and somewhat unsafe, route — Sect. 7.3. Fortunately, more familiar looking foothills come into sight, soon — Sect. 7.4. As one reaches them, one starts to feel ones fatigue — Sect. 7.5 — and one decides to take another short rest. Finally, one undertakes the last part of the way back to the valley (loosing the canonical path) — Sect. 7.6. One is longing for drinks and a good night’s sleep — Section 8 and preface.

Perhaps, the analogy sketched here is not entirely compelling, but it isn’t misleading.

It is probably superfluous to enter into a detailed description of the various chapters of these notes — the table of contents speaks for itself. But a few comments may be helpful.

Sections 1 and 2: Standard stuff — the experts should skip them (and reach the foothills by helicopter).

Section 3: An attempt to formulate some constraints on a tentative reconciliation of quantum theory with general relativity; (draws on ideas mostly due to other people). Reading recommended.

Sect. 4.1: An introduction to differential geometry for readers who are familiar with Pauli’s quantum mechanics of the non-relativistic, spinning electron. Hopefully useful.

Sect. 4.2: Good, old Lie group theory (put in clothes that physicists may like). Experts should skip it.

Sect. 4.3: A brief “tour d’horizon” of quantum theory, supersymmetry and geometry; (“global supersymmetry . . . is just another name for the differential topology and geometry of . . .spaces”). Should be clarifying.

Sect. 5.1: Some basic material on non-commutative geometry, according to Connes.

Sect. 5.2: The Riemannian aspect of non-commutative geometry and connections to global supersymmetry. A classification of geometries in terms of supersymmetries and broken supersymmetries. Some algebraic topology.
Sect. 5.3: Group actions on geometric spaces, BRST cohomology and target space supersymmetry; (“the air is getting thinner”).

Section 6: Analyzing some examples is all-important, in order to understand the general theory. (The non-commutative torus is, perhaps, the simplest non-trivial example; further examples appear in Sects. 7.5 and 7.6.)

Sect. 7.1: See Green-Schwarz-Witten, volume I.

Sect. 7.2: Some sections from Green-Schwarz-Witten, narrated in a, perhaps, somewhat personal way. Connections between the material in Section 5 and superstring theory are described. (Emphasis on Schwinger-Dyson equations for string Green functions.)

Sect. 7.3: A brief look into the future; (“anything goes” — Paul Feyerabend).

Sect. 7.4: Ground states (“vacua”) of superstring theory are described by certain two-dimensional (super-)conformal field theories. This section provides a short introduction to two-dimensional, local quantum field theory and (super-)conformal field theory and explains connections between conformal field theory and group theory.

Sect. 7.5: Reflections on the question what a conformal field theory describing a ground state of string theory teaches us about the geometry of space-time (more precisely, of “internal space”). An attempt to view conformal field theories as quantum theories describing “loop space geometry”.

Sect. 7.6: Tools to explore the topology and geometry of target spaces of superconformal field theories; an example (WZNW).

Sects. 7.4 – 7.6 could (or, perhaps, should) have been the core sections of these notes had the authors not started to feel their fatigue — nevertheless hopefully useful reading; (and then it will be time for the drinks).

Section 8: Conclusions are mostly left to the reader.

I wish to apologize for the shortcomings and imperfections of these notes and the many typos that may have escaped our attention. But we were really running out of time.

Much of the material in these notes is inspired by the work of A. Connes and the work of E. Witten and of their followers. Our efforts have been motivated by our desire to try to understand some of their work and to point to connections between their approaches. We have drawn on results and ideas of many other colleagues — too many to mention them all. Our line of thought is somewhat related to that of A. Jaffe and collaborators.

We should like to explicitly acknowledge our indebtedness to our collaborators, A.H. Chamseddine, G. Felder and K. Gawędzki. Had we not had the privilege of their cooperation and help we would have little to report here!

We are also grateful to many colleagues and participants of the school for most valuable discussions. We acknowledge countless lively discussions with S. Mukhanov.

We are indebted to A. Connes for sending us much of his work prior to publication and for various useful comments.

We thank A. Schultze for expert typing of the manuscript.

I am very grateful to A. Connes and K. Gawędzki for having invited me to participate as a student and to lecture at their school and for their most generous patience.

Jürg Fröhlich, May 1997
1 The classical theory of gravitation

In this section, we present a brief summary of how classical physics treats space and time, space-time geometry and its interrelation with gravitation. In physics, the geometry of space-time is an object of experimental exploration and physical modeling. Points in space-time are identified with (the location of) events, time-like curves with the world lines of observers or material objects. An observer gathers information about events or objects by recording light signals emitted from such events or objects and reaching her from her past light cone. Information between distant observers is exchanged with the help of signals consisting of electromagnetic waves. The dynamics of such signals is described by classical Maxwell theory. In relativistic physics it is assumed that information can never be exchanged with a velocity exceeding the velocity of light. Thus, it is assumed to be impossible to exchange information between space-like separated observers. This feature implies a fundamental unpredictability of the future in classical relativistic physics (observers can receive information, at best, about events inside their past light cone and hence, for most cosmologies, they can never gather complete information about “initial conditions” prescribed on some space-like Cauchy surface, because there do not usually exist Cauchy surfaces completely contained inside the past light cone of any observer. As a consequence, the maximal amount of information available, in principle, to an observer does not enable her to predict her own future with certainty).

Within classical physics, space-time is described as a four-dimensional Lorentzian manifold with certain good properties: It should admit a causal orientation (global distinction between the past and the future is possible); there should not exist any closed time-like geodesics (no “Gödel universes”); space-time singularities should be hidden behind horizons (“cosmic censorship hypothesis”).

In exploring space-time geometry, one assumes that one can detect signals causing arbitrarily small perturbations of the energy-momentum tensor (the “recoil” of signals on space-time geometry can be neglected). Among various forms of matter, one assumes the existence of point particles. An event is the emission or reflection of an electromagnetic wave by a point particle. One assumes that, with the help of arbitrarily weak signals, one can determine the state of a point particle arbitrarily precisely. This is based on the assumptions that arbitrarily precise watches are available and that the wavelength of an approximately monochromatic electromagnetic wave train can be measured arbitrarily precisely (within a space-time region so small that the deviation of its geometry from Minkowski space geometry is insignificant).

All these (fundamentally unrealistic) idealizations lead to the concept of space-time as a Lorentzian manifold with properties as described above.

According to the principle of general covariance, fundamental laws of nature should take the form of equations between tensor fields on the space-time manifold. According to the equivalence principle, it must be possible, locally in a small vicinity of a space-time point \( p \), to construct coordinate functions \( x^\mu \) vanishing at \( p \) and such that the space-time metric \( g_{\mu\nu}(p) \) at the point \( p \) is given by the Minkowski metric

\[
(\eta_{\mu\nu}) = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}
\]
and such that the Christoffel symbols vanish at \( p \). In these normal coordinates, the differential laws describing the dynamics of matter and radiation in an infinitesimal neighborhood of the point \( p \) are assumed to have the form known from special-relativistic physics.

Let \( T_{\mu\nu} \) denote the energy-momentum tensor of matter (including point particles, the electromagnetic field, etc.). Let \( R_{\nu\lambda\sigma}^\mu \) denote the Riemann curvature tensor, \( R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} \) the Ricci tensor, and \( r = R_{\mu}^\mu \) the curvature scalar. The Einstein tensor is defined by

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} r .
\]

As a consequence of the Bianchi identities, the covariant divergence of \( G_{\mu\nu} \) vanishes. The covariant divergence of the energy-momentum tensor \( T_{\mu\nu} \) vanishes, too (for Lagrangian models of matter). Thus, the Einstein-Hilbert field equation

\[
G_{\mu\nu} = \kappa T_{\mu\nu}
\]

is meaningful; \( \kappa \) is Newton’s constant and we choose units such that \( \hbar = c = 1 \). Simple dimensional analysis shows that \( \kappa \) has dimensions of length\(^{d-2} \), where \( d \) is the dimension of space-time: \( \kappa = l_P^{d-2} \), where \( l_P \) is the Planck length (\( l_P \approx 10^{-33} \text{ cm} \), or \( l_P^{-1} \approx 10^{19} \text{ GeV} \)).

It is well known that, in the limit of weak gravitational fields and for material objects with relative velocities small compared to the velocity of light, eqs. (1.2) formally reproduce the Newtonian theory of gravitation.

While, for suitably chosen initial conditions, eqs. (1.2) may have global solutions (see [1] for an important example) and hence they may express deterministic laws of nature, this does not imply that a localized observer can gather enough information to predict the future (as discussed above). This is a basic difference between non-relativistic and relativistic physics.

If \( T_{\mu\nu} \) is calculated for a gas of very light point particles then the equation \( \nabla_{\mu} T^{\mu\nu} = 0 \) implies an equation of motion for point particles: The world lines of point particles by its arc length (proper time) \( \tau \), the differential equation for a geodesic is

\[
\frac{d^2 x^\mu (\tau)}{d\tau^2} + \Gamma^\mu_{\nu\lambda} (x (\tau)) \frac{dx^\nu (\tau)}{d\tau} \frac{dx^\lambda (\tau)}{d\tau} = 0 ,
\]

where

\[
\Gamma^\mu_{\nu\lambda} (x) = \frac{1}{2} g^{\mu\sigma}(x) \left( \frac{\partial g_{\nu\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\lambda} - \frac{\partial g_{\nu\lambda}}{\partial x^\sigma} \right)
\]

are the Christoffel symbols.

Eqs. (1.3) can be derived from an action principle. The action is given by

\[
S(x(\cdot)) = \frac{1}{2} \int g_{\mu\nu} (x (\tau)) \frac{dx^\mu (\tau)}{d\tau} \cdot \frac{dx^\nu (\tau)}{d\tau} d\tau
\]

with the constraint

\[
g_{\mu\nu} (x (\tau)) \frac{dx^\mu (\tau)}{d\tau} \frac{dx^\nu (\tau)}{d\tau} = 1 ,
\]

(for massive test particles).
Unfortunately, the action (1.4) is not reparametrization-invariant (\(\tau\) is arc length). This problem can be cured by considering the actions

i) \[ S_{\text{NG}}(x(\cdot)) = \int ds = \int \sqrt{g_{\mu\nu}(x(\tau)) \frac{dx^\mu(\tau)}{d\tau} \frac{dx^\nu(\tau)}{d\tau}} \, d\tau , \]

where “NG” stands for Nambu-Goto, or

ii) \[ S_{\text{P}}(x(\cdot), h(\cdot)) = \frac{1}{2} \int g_{\mu\nu}(x(\tau)) \frac{dx^\mu(\tau)}{d\tau} \frac{dx^\nu(\tau)}{d\tau} h(\tau)^{-1/2} d\tau 
+ \frac{\mu^2}{2} \int h(\tau)^{1/2} d\tau , \]

(1.6)

where \(h(\tau)d\tau^2\) is an arbitrary metric on parameter space, i.e., on an interval \(I \subset \mathbb{R}\), and \(\mu > 0\) is a parameter. Here “P” stands for (Deser-Zumino-)Polyakov. Minimizing \(S_{\text{P}}\) with respect to \(h(\tau)\) yields the Euler-Lagrange equation

\[ h(\tau) = \frac{1}{\mu^2} g_{\mu\nu}(x(\tau)) \frac{dx^\mu(\tau)}{d\tau} \frac{dx^\nu(\tau)}{d\tau} , \]

(1.7)

or, for \(h(\tau) \equiv 1\),

\[ \tau = \frac{1}{\mu^2} \times \text{arc length} . \]

Upon using (1.7), the Euler-Lagrange equations obtained by minimizing (1.6) with respect to \(x(\tau)\) reproduce equation (1.3). The action (1.4) can be obtained from (1.6) by “fixing the gauge” \(h(\tau) \equiv 1\) (which destroys reparametrization invariance).

By observing the motion of a gas of test particles of very small mass with the help of electromagnetic waves, an observer can reconstruct the geometry of space-time regions contained in his past light cone. For example, he can measure the components \(R_{\mu00\nu}\) of the Riemann curvature tensor by studying a correspondence of geodesics (world lines of a gas of test particles). The Jacobi field \(n^\mu\) pointing from one geodesic in the correspondence to an infinitely close one satisfies the differential equation

\[ \frac{d^2 n^\mu(\tau)}{d\tau^2} = R_{\mu00\nu}(x(\tau)) \, n^\nu(\tau) . \]

The r.s. describes tidal forces whose observation apparently permits to measure \(R_{\mu00\nu}\).

It is well known how to extend the theory to systems of charged point particles moving through an external electromagnetic field.

All this is very beautiful; but the theory is plagued with inconsistencies. For example, it turns out to be impossible to take the concept of a point particle of positive mass (and non-zero charge) literally. It would lead to divergences and a-causal behaviour. But quite apart from such mathematical inconsistencies, it is impossible to describe matter and radiation by classical physical theories when “microscopic scales” are involved because, on such scales, their quantum mechanical nature becomes apparent. (When the action of the trajectory of a point particle is comparable to Planck’s constant \(\hbar\), its quantum-mechanical nature cannot be neglected.)
A fundamental problem of present-day physical theory is to reconcile (some form of) the quantum theory of matter and radiation with (some form of) Einstein’s relativistic theory of space-time and gravitation. To understand what the problem is, we shall briefly recapitulate some basic features of (non-relativistic) quantum theory and then explain in which way they are incompatible with general relativity.

2 (Non-relativistic) quantum theory

Quantum theory was born from the study of systems of harmonic oscillators. The first such system was black-body radiation corresponding to harmonic oscillations of electromagnetic waves in a cavity. Planck and Einstein found the rules of “quantization”. In his theory of the specific heat of crystals, Einstein showed that the same rules of quantization must also be applied to harmonic oscillations of material media, in order to reach agreement with experimental data. De Broglie extended these ideas to material particles by assuming that such particles have wave-like properties. The rules of quantization were eventually extended to apply to a rather general class of Hamiltonian systems with finitely many degrees of freedom and to systems of infinitely many oscillators with very weak anharmonicity. There is hardly any doubt that they apply to small (essentially harmonic) oscillations of the gravitational field (space-time metric) around a classical background field. In every example where a non-linear Hamiltonian system with infinitely many degrees of freedom is quantized, according to the standard rules, mathematical difficulties in the quantum theory can be traced to the fact that an arbitrary number of degrees of freedom can be localized in an arbitrarily small space-time region; or, in other words, that the number of possible events localized in an arbitrarily small space-time region is arbitrarily large. This suggests that there may be something wrong with the idea of space-time as a classical, smooth Lorentzian manifold when it comes to describing quantum mechanical events in very tiny regions of space-time. A considerable part of these notes is devoted to trying to find out what may go wrong and what might be done to cure the problem. (Of course, it won’t be solved here!)

We continue with a short and standard recapitulation of quantum mechanics. We start with a simple physical system consisting of a charged, non-relativistic particle interacting with the electromagnetic field. Historically, the physics of this system was explored on the basis of the following two ingredients:

(A) Classical, Newtonian mechanics of the particle and Maxwell’s theory of the electromagnetic field (which, together, form an infinite-dimensional Hamiltonian system).

(B) The theory of photons, due to Planck and Einstein, with the relations

\[ E = h\nu, \quad p = h/\lambda, \]

where \( h \) is Planck’s constant, \( E \) and \( p \) denote the energy and the momentum of a photon, i.e., of an electromagnetic field oscillator of frequency \( \nu \) and wave length \( \lambda = c/\nu \).

Unfortunately, these two ingredients are incompatible. Here is what goes wrong when one tries to combine (A) and (B) naively: The state of a charged particle at a given time is described by its position \( \vec{x} \) and its momentum \( \vec{p} \), the one of the electromagnetic field
by specifying the magnetic and the electric field on a space-like slice corresponding to, for example, a fixed time (in the rest frame of the particle). We may attempt to measure the state \( \vec{x}, \vec{p} \) of the particle as follows:

1. We turn on a homogeneous magnetic field in a region of space where we suspect to find the particle. Due to the Lorentz force, the trajectory of the particle is bent. If we know the electric charge of the particle and the velocity of light and have measured the strength of the magnetic field, we can (according to (A)) determine the momentum \( \vec{p} \) of the particle by measuring the curvature radius of its trajectory (which, incidentally, necessitates measuring the position of the particle at least three distinct times, or measuring the electromagnetic radiation caused by the accelerated motion of the particle).

2. We measure the position, \( \vec{x} \), of the particle by e.g. shining light into the region where we suspect to find the particle and detect light scattered by the particle with the help of a “Heisenberg microscope”. In studying the interaction of the particle with a light wave we use relations (B) and the conservation of total energy and momentum.

Let us suppose that, after having performed measurements (1) and then (2), we know the position \( \vec{x} \) and the momentum \( \vec{p} \) within a precision of \( \Delta x^j, \Delta p_j, j = 1, 2, 3; (x^j \text{ is the } j\text{th component of } \vec{x} \text{ in a Cartesian coordinate system}) \). Then \( \Delta x^j \) and \( \Delta p_j \) are constrained by Heisenberg’s uncertainty relations

\[
\Delta x^j \Delta p_j \geq \frac{\hbar}{2}, \quad j = 1, 2, 3, \tag{2.2}
\]

as some exceedingly well-known arguments show. Many different gedanken– and real experiments teach us that (2.2) is valid independently of what the tools used to measure \( \vec{x} \) and \( \vec{p} \) are. Similarly, when one attempts to measure the electric and magnetic field averaged over a small region \( \mathcal{O} \) of space-time by studying e.g. their influence on the motion of charged particles, whose positions and momenta are known only to an accuracy constrained by (2.2), one finds, according to Bohr and Rosenfeld, that the accuracies, \( \Delta E_\mathcal{O}, \Delta B_\mathcal{O} \), of these field measurements are constrained by

\[
\Delta B_\mathcal{O} \cdot \Delta E_\mathcal{O} \geq \hbar \text{const}_\mathcal{O}. \tag{2.3}
\]

Measuring an electromagnetic field in a space-time region through which a charged particle travels will thus create an uncertainty in the force acting on the charged particle.

Quantum mechanics is developed on the basis of the postulate that the uncertainty relations (2.2) and (2.3) are fundamental and hold independently of how the state of the system is measured.

It is useful to recast the discussion just presented in a more abstract context: We consider a classical Hamiltonian system, conveniently one with only finitely many degrees of freedom. We suppose that the phase space \( \Gamma \) of the system is the cotangent bundle \( T^*M \) of a smooth manifold \( M \), interpreted as the configuration space of the system. Phase space \( \Gamma \) is equipped with a symplectic 2-form \( \omega \). If \( U \) is an open subset of \( M \) over which the cotangent bundle is trivial, \( T^*U \simeq U \times \mathbb{R}^n \) (where \( n \) is the dimension of \( M \)), then one can choose coordinates \( q^1, \ldots, q^n \) in \( U \) and extend them to Darboux coordinates \( q^1, \ldots, q^n, p_1, \ldots, p_n \) for \( T^*U \) such that

\[
\omega = \sum_{j=1}^n dp_j \wedge dq^j. \tag{2.4}
\]
A state of the system in $T^*U$ is a point $(q, p) \in T^*U$, where $q = (q^1, \ldots, q^n)$ is interpreted as a configuration space position and $p = (p_1, \ldots, p_n)$ as momentum. The symplectic form $\omega$ is left invariant by symplectic diffeomorphisms of $\Gamma$. The dynamics of the system is specified by a Hamiltonian vector field $X_H$, where $H$ is a function on $\Gamma$; $X_H$ is defined by setting $\omega(X_H, Y) = Y(H) \equiv dH(Y)$, where $Y$ is an arbitrary vector field. The push forward of a Hamiltonian vector field under a symplectic diffeomorphism of $\Gamma$ is again a Hamiltonian vector field. “Observables” of the system are real-valued, continuous (or smooth) functions, $F$, on $\Gamma$. Every observable $F$ determines a Hamiltonian vector field $X_F$ (as above) and hence (locally) a one parameter group of symplectic diffeomorphisms. The algebra of observables with support in a closed subset $\Omega$ of $\Gamma$ is denoted by $\mathcal{F}(\Omega)$; and $\mathcal{F} := \mathcal{F}(\Gamma)$.

In passing from classical Hamiltonian dynamics to quantum theory, one supposes that, in any real measurement of the state $(q, p)$ of the system that determines $q$ up to a precision of $\Delta q$ and $p$ up to a precision $\Delta p$, the uncertainty relations

$$\Delta q^j \cdot \Delta p_j \geq \frac{\hbar}{2}, \quad j = 1, \ldots, n,$$  

must hold. One concludes that it is impossible to determine the state of the system precisely and that, therefore, the classical concept of a state is not strictly meaningful! It follows that the elements of $\mathcal{F}$ cannot be the observables of the system, because they separate points of $\Gamma$. Furthermore, one concludes that if $\Omega$ is a subset of $\Gamma$ of finite symplectic volume, $\text{vol}_\omega(\Omega) < \infty$, then, by the uncertainty relations (2.5), the number $N_\Omega$ of states of the system “located” in $\Omega$ that can be resolved by real measurements must be bounded by

$$N_\Omega \lesssim \frac{\text{vol}_\omega(\Omega)}{\hbar^n}. \quad (2.6)$$

Inequality (2.6) suggests that observables measurable in $\Omega$ should form an “essentially finite-dimensional” algebra, and hence $\mathcal{F}(\Omega)$ must be deformed to an algebra $\mathcal{F}_\hbar(\Omega)$ with this property.

An admissible quantization of the system must respect inequalities (2.5) and (2.6). In order to construct a quantization, one chooses an integrable polarization of $\omega$. A natural choice, in our context, is the vertical polarization for which configuration space is given by $M$. In the following, we only consider this choice. In order not to get stuck with technicalities, we temporarily assume that $M$ is smooth, compact, connected and simply connected. (For example in connection with quantum statistics, $\theta$-vacua, winding modes, etc., it is actually important to consider configuration spaces $M$ which are not simply connected or not even connected. But we postpone this issue.)

Next, we choose a Riemannian metric $g = (g_{\mu\nu})$ on $M$. We denote by $d\text{vol}_g$ the Riemannian volume form and by $\Delta_g$ the Laplace-Beltrami operator on smooth functions on $M$ associated with the metric $g$. We define

$$\mathcal{H} = L^2(M, d\text{vol}_g) \quad (2.7)$$

to be the Hilbert space of square-integrable functions on $M$. The operator $\Delta_g$ is essentially self-adjoint on the dense subspace of smooth functions in $\mathcal{H}$.

We also define a deformation $\mathcal{F}_\hbar$ of the algebra $\mathcal{F}$ of functions on $\Gamma$ as follows: Let $f \in C^\infty_0(\mathbb{R})$ be an arbitrary, smooth function on $\mathbb{R}$ of compact support. Since $-\Delta_g$ defines a positive, self-adjoint operator, $f(-\Delta_g)$ is well defined (by the functional calculus). The
operator $-\triangle_g$ defines a one parameter group, $\alpha_\tau$, $\tau \in \mathbb{R}$, of $^*$-automorphisms of the algebra $B(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ by setting

$$\alpha_\tau(A) := e^{-i\hbar\triangle_g} A e^{i\hbar\triangle_g}.$$  \hfill (2.8)

A reasonable definition of a deformation $\mathcal{F}_\hbar$ of the algebra $\mathcal{F}$ of continuous functions on $\Gamma$ is to define $\mathcal{F}_\hbar$ to be the smallest $C^*$-algebra generated by

$$\{\alpha_\tau(a), f(-\triangle_g) \mid a \in C(M), \tau \in \mathbb{R}, f \in C_0^\infty(\mathbb{R})\},$$ \hfill (2.9)

invariant under the $^*$-automorphism group $\alpha_\tau$. (Here and in the following, algebras of operators are defined over the field of complex numbers, unless stated otherwise. “Observables” will always correspond to self-adjoint elements of certain operator algebras.) The algebra $\mathcal{F}_\hbar$ can be thought of as the “algebra of functions over quantum phase space”. It contains the algebra

$$\mathcal{A} = C(M)$$ \hfill (2.10)

of complex-valued, continuous functions of $M$ as a maximal abelian $C^*$-subalgebra. For some class of compact regions $\Omega \subset \Gamma$, one can define algebras $\mathcal{F}_\hbar(\Omega)$ satisfying a suitable variant of the bound (2.6) in an obvious way.

It is useful to describe a second approach to constructing $\mathcal{F}_\hbar$: Let the Hilbert space $\mathcal{H}$ be as in (2.7). Given a diffeomorphism $\varphi$ of $M$, we define a unitary operator $U_\varphi$ on $\mathcal{H}$ by setting

$$(U_\varphi \psi)(x) := \sqrt{\varphi^* d\text{vol}_g(x)} \psi(\varphi^{-1}(x)),$$ \hfill \hfill (2.12)

The unitarity of $U_\varphi$ follows from the invertibility of $\varphi$ and the quasi-invariance of $d\text{vol}_g$ under diffeomorphisms. We define

$$\mathcal{U} := \{U_\varphi \mid \varphi \in \text{Diff } M\}$$

and choose $\mathcal{F}_\hbar$ to be the smallest $C^*$-algebra generated by $\mathcal{U}$ and by $C(M)$. If $M$ consists of a finite number, $n$, of points one easily shows that $\mathcal{F}_\hbar = \mathbb{M}_n(\mathbb{C})$.

A quantization of a classical Hamiltonian system with phase space $\Gamma = T^* M$, ($M$ compact, smooth, connected and simply connected) satisfying the requirements expressed in inequalities (2.5) and (2.6) consists in choosing spectral data of e.g. the form

$$(\mathcal{F}_\hbar, \mathcal{A} = C(M), \mathcal{H} = L^2(M, d\text{vol}_g), \triangle := -\triangle_g).$$ \hfill (2.11)

Dynamics is specified by choosing a self-adjoint operator $H$ densely defined on $\mathcal{H}$ with the properties:

(i) $e^{itH/\hbar} a e^{-itH/\hbar} \in \mathcal{F}_\hbar$ for $a \in \mathcal{F}_\hbar$ and for all $t \in \mathbb{R}$, i.e., the unitary operators $e^{itH/\hbar}$ determine a one-parameter group of $^*$-automorphisms of $\mathcal{F}_\hbar$; and

(ii) in the limit $\hbar \downarrow 0$, some classical Hamiltonian dynamics on $\Gamma = T^* M$ is recovered (in a standard sense that we do not make precise here; “quantization” is a “deformation” of classical, Hamiltonian mechanics).

Since nature is intrinsically quantum-mechanical, it is, perhaps, more interesting to ask how, from spectral data of quantum mechanics, one can reconstruct e.g. the topology
and geometry of configuration space $M$, rather than to pursue the question what we mean by the quantization of a classical Hamiltonian system. (Of course, reconstructing $M$ may not necessarily directly teach us anything about physical space, but something about configuration space.) It is interesting to ask which data of quantum theory suffice to reconstruct configuration space $M$, together with a Riemannian metric $g$ on $M$. The complete data are $(\mathcal{F}, \mathcal{A}, \mathcal{H}, \triangle)$, as in (2.11), where $\mathcal{F} \equiv \mathcal{F}_h$, and the subscript “$h$” will be omitted from now on. Of course, these data are redundant, because, knowing $\mathcal{A}, \mathcal{H}$ and $\triangle, \mathcal{F}$ is determined (by the construction described above). We propose to view $(\mathcal{F}, \mathcal{A}, \mathcal{H}, \triangle)$ as abstract spectral data, where

1. $\mathcal{H}$ is a separable Hilbert space;

2. $\triangle$ is a positive, self-adjoint operator on $\mathcal{H}$;

3. $\mathcal{A}$ is an abstract abelian $C^*$-algebra with the following properties:

   a. $\mathcal{A}$ has a faithful *-representation, $\pi$, on $\mathcal{H}$;

   b. $\mathcal{A}$ contains a subalgebra $\mathcal{A}$ dense in $\mathcal{A}$ in the $C^*$-norm of $\mathcal{A}$ such that the operator

   $\frac{1}{2} \left( \triangle \pi(a)^2 + \pi(a)^2 \triangle \right) - \pi(a) \triangle \pi(a)$

   is bounded in norm for arbitrary $a \in \mathcal{A}$;

   c. $\mathcal{F}$ is constructed from $\mathcal{A}$ and $\triangle$ as in (2.9), and $\mathcal{A}$ is maximal abelian in $\mathcal{F}$.

Given spectral data satisfying these properties, we may ask the following questions:

(a) Does the pair $(\mathcal{H}, \triangle = -\triangle_g)$ determine the manifold $M$ and its geometry? The answer is, in general, no: one cannot “hear the shape of a drum” [2]. However, certain properties of $M$ are determined by $(\mathcal{H}, \triangle)$. For example, according to a celebrated result due to H. Weyl, the (asymptotics of the) spectrum of $\triangle$ determines the dimension of $M$ and its Riemannian volume. Furthermore, using ideas that physicists know from the theory of quantum-mechanical tunneling, one can show [3] that

$\frac{1}{4} h_M^2 \leq \lambda_1(M) \leq C (\delta h_M + h_M^2)$,

where $h_M = h_M(g)$ is Cheeger’s isoperimetric constant of $(M, g)$, and $\lambda_1(M) = \lambda_1(M, g)$ is the smallest non-zero eigenvalue of $-\triangle_g$, $C$ is a universal constant, and $\delta$ is a constant depending on the diameter of $M$. Yet, even when dim $M = 2$, there are isospectral manifolds that are not isometric [2].

(b) Does $\mathcal{A}$ determine $M$ and its geometry? A famous theorem, due to Gel’fand (see e.g. [4]), says that the space of characters of an abelian unital $C^*$-algebra $\mathcal{A}$ (the “spectrum” of $\mathcal{A}$) is a compact Hausdorff space $K$ with the property that $\mathcal{A} = C(K)$ (the algebra of complex-valued, continuous functions on $K$). The space $K$ encodes the properties of $M$ when $M$ is viewed as a compact Hausdorff space, but it does not tell us anything about e.g. a differentiable structure on $M$, a geodesic distance on $M$, etc.
(γ) Do the data \((\mathcal{A}, \mathcal{H}, \triangle)\) determine the topology and Riemannian geometry of \(M\)? Here the answer is yes: \(\mathcal{A}\) determines \(M\) as a compact Hausdorff space. The algebra

\[
\hat{\mathcal{A}} = \left\{ a \in \mathcal{A} \mid \frac{1}{2} \left( \triangle a^2 + a^2 \triangle \right) - a \triangle a \leq \infty \right\},
\]

which is a norm-dense subalgebra of \(\mathcal{A}\), by assumption (b), can be interpreted as the algebra of \textit{Lipschitz-continuous functions} on \(M\): If \(\mathcal{A} = C(M)\) and \(\triangle = -\triangle_g\) then

\[
\frac{1}{2} \left( \triangle a^2 + a^2 \triangle \right) - a \triangle a = -g^\mu{}^\nu \left( \partial_\mu a \right) \left( \partial_\nu a \right),
\]

(2.13)

see [24]. A variant of an argument due to Connes [5] enables us to reconstruct the \textit{geodesic distance} on \(M\) determined by \(g\): Given two points, \(x\) and \(y\), in \(M\) (i.e., two characters of \(\mathcal{A}\)), the geodesic distance between \(x\) and \(y\) is given by

\[
\text{dist}_g(x, y) = \sup |a(x) - a(y)|,
\]

(2.14)

where the supremum on the r.s. of (2.14) is taken over all elements \(a\) of \(\hat{\mathcal{A}}\) with the property that

\[
\frac{1}{2} \left( \triangle a^2 + a^2 \triangle \right) - a \triangle a \leq 1.
\]

(2.15)

It is not hard to introduce higher-order polynomials in \(\triangle\) and elements of \(\mathcal{A}\), in order to test whether \(M\) is smooth. (It is not surprising that, on a classical manifold, one can usually define a notion of \textit{Lipschitz-continuous functions}, because, according to a theorem due to D. Sullivan, a topological manifold of dimension \(\neq 4\) is automatically \textit{Lipschitz}.)

Connes’ theory of non-commutative geometry [5] starts from the idea (among other ideas) to view abstract spectral data \((\mathcal{A}, \mathcal{H}, \triangle)\), where \(\mathcal{A}\) is a \(*\)-algebra of bounded operators on the separable Hilbert space \(\mathcal{H}\), and \((\mathcal{A}, \mathcal{H}, \triangle)\) have properties (a) and (b), as a starting point for “non-commutative geometry”.

In the study of quantum-mechanical systems with \textit{infinitely many degrees of freedom} (quantum field theory) one often encounters a problem in trying to make sense of \(\mathcal{A}\); but an analogue of the \textit{non-commutative} algebra \(\mathcal{F}\) is still meaningful. Recall that \(\mathcal{F}\) may be interpreted as the “algebra of functions” on “quantum phase space”; historically, the first (and a paradigmatic) example of a \textit{non-commutative space}. This motivates us to ask the question:

(δ) Under which additional hypotheses do the data \((\mathcal{F}, \mathcal{H}, \triangle)\), where \(\mathcal{F}\) is a \(\mathcal{C}^*\)-algebra faithfully represented on the separable Hilbert space \(\mathcal{H}\) and \textit{invariant} under the \(*\)-automorphism group \(\alpha_\tau\) defined by

\[
\alpha_\tau(A) := e^{i\tau \triangle} A e^{-i\tau \triangle},
\]

\(A \in B(\mathcal{H}), \tau \in \mathbb{R}\), determine the topology and Riemannian geometry of a classical manifold \(M\)? Suppose we know that \((\mathcal{F}, \mathcal{H}, \triangle)\) are as in (2.7–9) (i.e., they come from a classical Riemannian manifold \(M\), and the operator \(\triangle\) is identified with \(\triangle_g\), how do we reconstruct \(M\) and \(g\) just from \((\mathcal{F}, \mathcal{H}, \triangle)\)?

These are important questions to which we don’t know complete answers, at present. They deserve the attention of mathematicians, as will become apparent in subsequent sections. We shall describe some tools that may lead to satisfactory answers. But we
anticipate one important observation: In general, topologically distinct classical configuration spaces $M_1, M_2, \ldots$ may lead to the same spectral data $(\mathcal{F}, \mathcal{H}, \Delta)$ – “$T$-duality”.

Let us briefly summarize the main points of this section.

(I) Nature is quantum-mechanical. Quantum theory, described in terms of spectral data such as $(\mathcal{A}, \mathcal{H}, \Delta)$ in (1)-(3) above, enables one to reconstruct a manifold $M$, interpreted as configuration space, equipped with a Lipschitz structure and with a Riemannian metric $g$. The manifold $M$ has, in general, nothing to do with (a Cartesian product of several copies of) physical space, unless we study systems of non-relativistic point particles.

(II) “Observables” of a quantum-mechanical system are self-adjoint elements of the algebra $\mathcal{F}$ (of “functions” on “quantum phase space”). Whereas in classical Hamiltonian mechanics observables generate one-parameter groups of symplectic diffeomorphisms, “observables” of a quantum-mechanical system generate one-parameter groups of unitary operators on the Hilbert space $\mathcal{H}$. Pure states in classical mechanics are points in phase space. In quantum theory, they are unit rays in $\mathcal{H}$ (i.e., points in a usually infinite-dimensional complex projective space).

(III) In Hamiltonian mechanics, “symmetries” are symplectic diffeomorphisms of phase space; in quantum theory “symmetries” are unitary operators on a Hilbert space $\mathcal{H}$ (defining *-automorphisms of the algebra $\mathcal{F}$). Dynamics is specified by a Hamiltonian vector field in classical mechanics; in quantum theory, it is specified by a self-adjoint operator $H$, the Hamiltonian (with the property that the unitary operators $e^{itH/\hbar}, t \in \mathbb{R}$, determine a one-parameter group of *-automorphisms of $\mathcal{F}$).

(IV) Let $\mathcal{H}$ be the Hilbert space of state vectors of a quantum-mechanical system, and let $H$ denote its Hamiltonian. For a unit ray $\psi \in \mathcal{H} (\|\psi\| = 1)$, we define

$$\triangle E_\psi := \| (H - \langle \psi, H \psi \rangle) \psi \|$$

(2.16)

to be the “energy-uncertainty” in the state vector $\psi$. One can show that the time $\triangle t_\psi$ it takes before the time evolution $e^{-itH/\hbar}\psi$ of $\psi$ “deviates substantially” from $\psi$ satisfies the uncertainty relation

$$\triangle E_\psi \cdot \triangle t_\psi \gtrsim \hbar .$$

(2.17)

For a precise version of (2.17) and generalizations and applications thereof (e.g. to the theory of resonances) see [6]. One may interpret $\triangle t_\psi$ as the “life time” of the state described by $\psi$.

Note that, in the present section, we have reviewed some aspects of non-relativistic quantum theory (leaving aside notions like that of a particle, of its quantum statistics, etc.) without taking into account gravitational effects predicted by the general theory of relativity. In the next section, we shall sketch some general conclusions drawn from an attempt to combine quantum theory with general relativity. Subsequently, we shall turn off the coupling between matter and the gravitational field again and ask what the quantum theory of non-relativistic point particles with spin (non-relativistic electrons, positrons and positronium) teaches us about differential topology and geometry of manifolds.
3 Reconciling quantum theory with general relativity: quantum space-time-matter

Physics as we know it, at present, is founded on two pillars:

(A') The analysis of causal sequences of events in a classical space-time, where classical space-time is a Lorentzian manifold (with properties as described in Section 1), and the geometry of space-time coupled to matter is described by Einstein’s field equations

\[ G_{\mu\nu} = \kappa T_{\mu\nu} . \]

(B') Quantum Theory: Localized events in space-time are caused by the radiative decay of unstable, localized states of matter. Matter and radiation are quantum-mechanical. (The “rules of quantization” must be applied to all degrees of freedom of a physical system evolving according to some Hamiltonian dynamics: particles, vibrations of material media, field oscillators, . . . , gravitational waves, and hence, ultimately, to space-time geometry.)

In their present form, these two pillars are incompatible. Fortunately, space-time appears to be classical down to distance scales of the order of the Planck scale, \( l_p \). From a purely pragmatic point of view, merging space-time structure with quantum theory in a consistent, unified theory is therefore not an urgent task. However, it is an important task from the point of view of logical consistency of physical theory.

To anticipate one conclusion of our analysis, we propose — and we do not claim to be original in this — that space-time should be viewed as a secondary, or derived structure, one that emerges from an underlying fundamental quantum theory of natural phenomena that treats space, time and matter on an equal footing and that, a priori, does not talk about space and time. Space-time is expected to be a feature of such a theory that only emerges in certain limiting regimes — just like classical physics can emerge from quantum physics in certain limiting regimes.

In attempting to fill this proposal with substance, one will observe that the concept of point-like events and point-particles is untenable and that, as a consequence, one needs a generalization of the notion of a classical space and of classical differential topology and -geometry. At this point, Connes’ theory of non-commutative geometry is suggestive of what we might want to look for, mathematically.

The question addressed in this section is: Why does a combination of quantum theory and general relativity force us to modify the concept of space-time as a classical Lorentzian manifold; what goes wrong with the idea of point particles and point-like events (i.e., ones localized in an arbitrarily small region of space-time)? In the following, we sketch some crude answer to this question, following ref. [7]; see also [8] for various details.

What we perceive as a localized event in space-time is always the decay of an unstable, localized state of a physical subsystem that is inherently quantum-mechanical.
The temporal duration of the event is denoted $\Delta t$, $d'$ denotes its maximal and $d''$ its minimal spatial extension. An observer is located in the forward light cone of the event (spatially far separated from the event). To be able to give meaning to the quantities $\Delta t$, $d'$ and $d''$, space-time (outside the event and around the observer) must be equipped with a metric. The observer applies the usual Heisenberg uncertainty relations of quantum theory to interpret the observed event.

(a) The energy uncertainty $\Delta E$ (defined e.g. as in (2.16)) is bounded below by

\[ \Delta E \gtrsim \frac{1}{\Delta t}, \]  

in units where $\hbar = 1$ and $c = 1$. We also invoke the standard uncertainty relation

\[ |\Delta \vec{p}| \gtrsim \frac{1}{d''}. \]  

Assuming that the motion of the center of mass of the event obeys the laws of the special theory of relativity, we conclude that if $d'' \lesssim \frac{1}{M}$, where $M$ is the rest mass of the event then

\[ \Delta E \gtrsim \frac{1}{d''}, \]  

which improves (3.1) if $d'' \ll \Delta t$.

(b) Suppose that $\Delta t \gg d' \approx d''$. Then the metric well outside the region where the event is localized is given, approximately, by the Schwarzschild metric (or a Schwarzschild-Newman metric if the event carries electric charge). The Schwarzschild radius (radius of the event horizon) is bounded by

\[ r_s \gtrsim \Delta E \cdot l_p^2. \]  

12
Next, suppose that $\Delta t \gg d' \gg d''$. Then the metric outside the event is given, approximately, by the Kerr(-Newman) metric describing an object with non-vanishing angular momentum $l$. Let $r'$ and $r''$ be the maximal and minimal spatial extensions of the event horizon, and let $l \lesssim \Delta Er'$. Then $\Delta E \lesssim \frac{d''}{l_p}$ and
\[
 r' \cdot r'' \approx \frac{l_p^2}{l_p} .
\]

(c) Suppose that Hawking is wrong, and a black hole is really black. Then decay products of the event will reach the observer only if
\[
d' \approx d'' \gtrsim r_S \gtrsim \Delta E \cdot \frac{l_p^2}{l_p} ,
\]
for the Schwarzschild metric. Combining (3.6) with (3.3), we conclude that
\[
d' \approx d'' \gtrsim l_p .
\]
Combining (3.6) with (3.1), we get that
\[
d' \Delta t \gtrsim l_p^2 .
\]
For the Kerr metric we use that $\Delta E \lesssim \frac{r''}{l_p}$ and hence, using (3.3) and the inequalities $d' \gg d'' \gtrsim r''$, we conclude that
\[
\frac{1}{d'} \lesssim \Delta E \lesssim \frac{d''}{l_p} ,
\]
hence
\[
d' \cdot d'' \gtrsim l_p^2 ,
\]
and, using that $\Delta E \gtrsim \frac{1}{\Delta t}$ and $d' > d''$, it follows that
\[
d' \cdot \Delta t \gtrsim l_p^2 .
\]
In all cases, we appear to conclude that if an event is not encased in a black hole then
\[
d' \cdot d'' \gtrsim l_p^2 \quad \text{and} \quad d' \cdot \Delta t \gtrsim l_p^2 .
\]
These are the uncertainty relations first proposed in [7].

Next, we assume that Hawking’s laws of black hole evaporation are right. If $d'$ and $d''$ denote the maximal and minimal spatial extension of a black hole then its mass is
\[
M \gtrsim \frac{d''}{l_p} .
\]
The Hawking temperature of the black hole is [9]
\[
kT = \frac{1}{8\pi l_p^2 M} .
\]
Elementary thermodynamics then implies that
\[
- \frac{dM}{dt} \approx \gamma \left( \frac{1}{d''} \right)^4 \cdot (d')^2 = \frac{\gamma}{(d'')^2} \left( \frac{d'}{d''} \right)^2
\]
\[
\approx \frac{\gamma}{l_p^2} M^{-2} \left( \frac{d'}{d''} \right)^2 .
\]
for a dimensionless constant $\gamma$. From (3.14) we obtain a bound on the life time of a radiating black hole

$$\Delta t \approx l_P^4 M^2 \left(\frac{d''}{d'}\right)^2 \Delta E,$$

(3.15)

with $\Delta E = \Delta M \approx M$ (provided the initial mass of the black hole is large compared to $\frac{1}{l_P}$). Using that $\Delta E \gtrsim \frac{1}{\Delta t}$ and $M \approx \frac{d''}{l_P}$, we conclude that

$$(\Delta t)^2 \left(\frac{d'}{d''}\right)^2 \gtrsim (d'')^2,$$

hence

$$\Delta t \cdot d' \gtrsim (d'')^2.$$  

(3.16)

Furthermore, by (3.1), (3.15) and since $d'' < d'$,

$$1 \lesssim \Delta t \cdot \Delta E \approx l_P^4 M^2 (\Delta E)^2 \approx l_P^4 M^4$$

and hence, using (3.12),

$$d'' \gtrsim l_P.$$  

(3.17)

In conclusion, we find that if a localized event can be interpreted as the evaporation of a black hole then, again,

$$d' \cdot d'' \gtrsim l_P^2 \quad \text{and} \quad \Delta t \cdot d' \gtrsim l_P^3.$$  

(3.18)

see [8]. From (3.12), (3.13) and (3.17) we also derive that

$$kT \approx \frac{1}{l_P}.$$  

(3.19)

(an upper bound for the Hawking temperature of a black hole). We also recall the expression, due to Bekenstein and Hawking, for the entropy of a black hole (in four space-time dimensions)

$$S = A/4l_P^2,$$

(3.20)

where $A$ is the area of the horizon of the black hole. This expression suggests that the number $N_A$ of distinct states of a black hole is bounded by

$$N_A \approx \exp \left( \text{const} A/l_P^{d-2} \right).$$  

(3.21)

In all these formulas, we do not pay attention to values of various dimensionless constants.

We should emphasize that our analysis is based on the assumption that uncertainty relations and Einstein’s field equations are valid down to scales comparable to the Planck scale. It can certainly not be excluded that quantum theory and the general theory of relativity are modified, in a more fundamental theory, in such a way that our analysis is invalidated. (This might be the case if one succeeded in constructing some asymptotically free quantum field theory of matter and the gravitational field; but there is no evidence, at present, that such a theory can be constructed.) Keeping the above warning in mind, we shall take the point of view that the bounds (3.18) on the extension of events in space and time, the bound (3.19) on the temperature of events and the bound (3.21) on the number of distinct states of a black hole are fundamental. Our derivation of these bounds follows
[7,8]; but see also [9,10,11]. A basic result in ref. [7] (the work that partly motivated [8]) is that the uncertainty relations (3.18) are compatible with the special theory of relativity (Poincaré covariance) on large scales.

On the basis of inequalities (3.18), one may argue that the number $N_\mathcal{O}$ of events or “excited modes” of matter localized inside an open region $\mathcal{O}$ of finite (metric) volume $\text{vol}_g(\mathcal{O})$ that can be distinguished, in principle, experimentally is bounded by

$$N_\mathcal{O} \lesssim \frac{\text{vol}_g(\mathcal{O})}{l_P^d}, \quad (3.22)$$

where $d$ is the dimension of space-time (and $d > 2$). Combining this bound with (3.21), we are tempted to conclude that the total number of distinct observations of events localized inside some open space-time region $\mathcal{O}$, with $\text{vol}_g(\mathcal{O}) < \infty$, is essentially bounded by $\exp(\text{const} \cdot \text{vol}_g(\mathcal{O})/l_P^d)$. If $\mathcal{A}_{l_P}(\mathcal{O})$ denotes the algebra of observables localized in $\mathcal{O}$, and $\text{vol}_g(\mathcal{O}) < \infty$, we may then argue that $\mathcal{A}_{l_P}(\mathcal{O})$ is “essentially finite-dimensional”:

$$\dim(\mathcal{A}_{l_P}(\mathcal{O})) \lesssim \exp\left(\text{const} \cdot \text{vol}_g(\mathcal{O})/l_P^d\right). \quad (3.23)$$

In particular, if space-time is foliated in compact space-like hypersurfaces of codimension 1 one might want to predict that the algebra of all local observables is (essentially) finite dimensional.

These conclusions are highly plausible, unless the coupling of modes of very high energies (comparable to or higher than the Planck energy) to the gravitational field becomes weak and tends to 0 as the energy increases to infinity. We propose to take them seriously as long as modes of energies $\gtrsim \frac{1}{l_P}$ remain unexcited.

In local relativistic quantum field theory [58], the local algebras $\mathcal{A}(\mathcal{O})$ are von Neumann factors of type III$_1$ (see e.g. [12]) and hence are genuinely infinite-dimensional (if $\mathcal{O}$ is, for example, a bounded open double cone). As a consequence, the bounds (3.22) and (3.23) are violated. However, if a local, relativistic quantum field theory describes a finite number of species of massive asymptotic particles then the number of states localized in an open region $\mathcal{O}$ of physical space, with $\text{vol}_g(\mathcal{O}) < \infty$, and with an energy $\leq \varepsilon \text{vol}_g(\mathcal{O})$, for an arbitrary $\varepsilon < \infty$, is expected to be bounded by $\exp(\text{const}_\varepsilon \cdot \text{vol}_g(\mathcal{O}))$; see [13] and refs. given there. The problem is that $\text{const}_\varepsilon \to \infty$, as $\varepsilon \to \infty$.

It is sometimes argued that, on the r.s. of (3.22) and (3.23), the dimensionless volume of $\mathcal{O}$ can be replaced by the dimensionless area of the boundary of $\mathcal{O}$ (“holographic principle” [14]). But the plausibility of this prediction is very limited (unless one considers a single black hole).

We draw the reader’s attention to the similarity between the bound (2.6) and the bound (3.22).

The bounds (3.18), (3.22) and (3.23) say that it is impossible to determine the location of an event or of some excited modes of matter in space and time arbitrarily precisely and suggest that, as a consequence, the concept of a space-time continuum is not strictly meaningful. The (universal) algebras $\mathcal{A}(\mathcal{O})$ of observables measurable in a bounded, open space-time region $\mathcal{O}$ provided by local relativistic quantum field theory cannot, ultimately, be the true local algebras. They must be deformed to algebras $\mathcal{A}_{l_P}(\mathcal{O})$ satisfying inequality (3.23) – just like the algebras $\mathcal{F}(\Omega)$, $\Omega \subset \Gamma$, of “observables” of classical mechanics must be deformed to the algebras $\mathcal{F}_\hbar(\Omega)$ of “observables” of quantum mechanics in such a way that the bound (2.6) holds. The deformation of classical physics to a fundamental
theory of space, time and matter thus involves two deformation parameters, $\hbar$ and $l_P$. The two deformations, in $\hbar$ and in $l_P$, are expected to solve the problem of singularities of space-time. For example, the evaporation of a black hole is not expected to result in a singularity, because space-time is non-commutative at the Planck scale.

We have reached satisfactory understanding of what is going on on the $\hbar$-axis (at $l_P = 0$) – quantum theory of matter in a classical space-time background – and on the $l_P$-axis (at $\hbar = 0$) – classical general relativity. But nature is found in the region $\hbar > 0$, $l_P > 0$. String perturbation theory is an attempt to understand something about nature by constructing an expansion in powers of some function of $l_P$ about $l_P = 0$. String theory is clever in that it treats matter quantum-mechanically in such a way that, on the tree level, some understanding of what is going on on the $l_P$-axis is automatically built into the theory.

In conventional quantum gravity, one attempts to construct an expansion in powers of $\hbar$ about $\hbar = 0$ (the $l_P$-axis), in the following way: One starts by viewing space-time as a classical manifold equipped with a metric, $g^c_{\mu\nu}$, that is a solution of the Einstein equation $G^c_{\mu\nu} = \kappa T^c_{\mu\nu}$ (the superscript “c” stands for “classical”). One then attempts to include small quantum corrections by setting $T_{\mu\nu} = T^c_{\mu\nu} + \Delta T_{\mu\nu}$, where $\Delta T_{\mu\nu}$ is a “small” operator-valued field. This forces one to also interpret $G_{\mu\nu}$ as operator-valued, which, in turn, entails that the coefficients of the connection $\nabla_\mu$ on the (co-)tangent bundle of the space-time manifold must be operator-valued, as well. But then vector fields and differential forms over space-time must be operator-valued, too. Hence the pairing of a one-form with a vector field, which should yield a function on space-time, will in general yield an operator-valued function. This suggests that space-time cannot be viewed as a classical manifold!

In trying to maintain locality (Einstein causality) of the quantum theory, one would like to imagine that the causal structure on space-time inherited from the metric $g^c_{\mu\nu}$ solving the classical equations $G^c_{\mu\nu} = \kappa T^c_{\mu\nu}$, provides an appropriate notion of locality for the quantum theory. This would be o.k. if the “quantized” metric $g_{\mu\nu}$ were conformally equivalent to $g^c_{\mu\nu}$ — i.e. $g_{\mu\nu} = \exp(\phi)g^c_{\mu\nu}$ for some quantum field $\phi$ —, which, in general, cannot be true if $d > 2$. The only way out appears to be to give up the idea of space-time as a classical manifold! In other words, the problem of quantum gravity is not, actually, a problem of calculating perturbations in quantum theory arising from gravitational interactions, or perturbations in general relativity arising from quantum-mechanical fluctuations, but to construct a two-parameter deformation, in $\hbar$ and $l_P$, of the laws of classical physics resulting in a non-commutative generalization of geometry.

Our discussion can be summarized by postulating that real microscopes cannot resolve a number of distinct events located in an open region of space-time of finite volume that would exceed the bound given in (3.22), and that they cannot be used to determine the location of an event in space-time with an accuracy violating (3.18). The bounds (3.18), (3.22) and (3.23) are assumed to be valid independently of how such an “Einstein-Heisenberg microscope” is built and operated. Intuitively, one would expect that a theory compatible with (3.18) and (3.22,23) had better be a quantum theory of “extended objects” (corresponding to the two deformation parameters $\hbar$ and $l_P$) that treats space, time and matter on an equal footing.

Before we try to describe aspects of such a theory, we shall return to the study of (non-relativistic) quantum theory of point particles with spin, setting $l_P = 0$. We propose to find out what it teaches us about the geometry of physical (Newtonian) space.
4 Classical differential topology and -geometry and supersymmetric quantum theory

In this section we describe an approach to differential topology and -geometry based on the quantum theory of non-relativistic point particles with spin, Pauli’s electron, positron and bound states thereof. The quantum theory of these particles exhibits supersymmetries. We show how the classification of different types of differential geometries can be derived from the classification of supersymmetries. Our approach is inspired by ideas in [15,16,17,5] and has appeared in [18,19]; another useful reference is [20].

Throughout this section, the recoil of matter on the gravitational field is neglected, and matter is thought to consist of non-relativistic point particles. We want to clarify what the quantum theory of such particles teaches us about the geometry of \textit{physical space} (time is a parameter). Our presentation follows the general ideas described in Section 2. But we shall consider the quantum theory of a single particle with \textit{spin}, and spin will turn out to play a fundamental role. The results of this section set the stage for a generalization of topology and geometry that enables us to study non-commutative spaces, as pioneered by Connes [5]. That generalization will be described in the next section. The tools described there are likely to be useful in exploring aspects of a theory, yet to be found, that unifies the quantum theory of matter with gravitation.

We start this section with a recapitulation of Pauli’s quantum theory of the non-relativistic electron with spin, generalized to arbitrary space dimension.

4.1 Pauli’s electron

Physical space is chosen to be a smooth, orientable, Riemannian spin manifold \((M,g)\) of dimension \(n\), where \((g_{ij})\) denotes the metric on the tangent bundle \(TM\) and \((g^{ij})\) the (inverse) metric on the cotangent bundle \(T^*M\). Let \(\Lambda^\bullet M = \bigoplus_k (T^\ast M)^\wedge k\) denote the bundle of completely anti-symmetric covariant tensors over \(M\). Let \(\Omega^\bullet (M)\) be the space of smooth sections of \(\Lambda^\bullet M\), i.e., of smooth differential forms on \(M\), and \(\Omega^\bullet_C (M) = \Omega^\bullet (M) \otimes \mathbb{C}\) its complexification. Since we are given a Riemannian metric on \(M\), \(\Omega^\bullet_C (M)\) is equipped with a Hermitian structure which, together with the Riemannian volume element \(d\text{vol}_g\), determines a scalar product \((\cdot,\cdot)_g\) on \(\Omega^\bullet_C (M)\). Let \(\mathcal{H}_{e-p}\) denote the Hilbert space completion of \(\Omega^\bullet_C (M)\) in the norm determined by \((\cdot,\cdot)_g\). Thus \(\mathcal{H}_{e-p}\) is the space of complex-valued, square-integrable differential forms on \(M\). This Hilbert space is \(\mathbb{Z}\)-graded,

\[
\mathcal{H}_{e-p} = \bigoplus_{k=0}^n \mathcal{H}_{e-p}^{(k)},
\]

where \(\mathcal{H}_{e-p}^{(k)}\) is the subspace of square-integrable differential forms of degree \(k\).

Given a one-form \(\xi \in \Omega^1(M)\), let \(X\) be the vector field corresponding to \(\xi\) by the equation

\[
\xi(Y) = g(X,Y),
\]

for any smooth vector field \(Y\). For every \(\xi \in \Omega^1_C(M)\), we define two operators on \(\mathcal{H}_{e-p}\):

\[
a^*(\xi) \psi = \xi \wedge \psi
\]
and

\[ a(\xi)\psi = X \rightarrow \psi , \quad (4.4) \]

for all \( \psi \in \mathcal{H}_{c-p} \). In (4.4), \( \rightarrow \) denotes interior multiplication. One easily checks that \( a^*(\xi) \)

is the adjoint of \( a(\xi) \) in the scalar product of \( \mathcal{H}_{c-p} \). Furthermore, one verifies that, for arbitrary \( \xi \) and \( \eta \) in \( \Omega^1_{\mathbb{C}}(M) \),

\[
\{ a(\xi), a(\eta) \} = \{ a^*(\xi), a^*(\eta) \} = 0, \quad (4.5)
\]

\[
\{ a(\xi), a^*(\eta) \} = g(\xi, \eta) , \quad (4.5)
\]

where \( \{ A, B \} := AB + BA \) denotes the anti-commutator of \( A \) and \( B \), and we use the symbol \( g \) to denote the (inverse) metric on \( T^*M \). Eqs. (4.5) are called canonical anti-commutation relations and are basic in the description of fermions in physics.

Next, for every real \( \xi \in \Omega^1(\mathbb{M}) \), we define two anti-commuting anti-selfadjoint operators \( \Gamma(\xi) \) and \( \bar{\Gamma}(\xi) \) on \( \mathcal{H}_{c-p} \) by

\[
\Gamma(\xi) = a^*(\xi) - a(\xi), \quad (4.6)
\]

\[
\bar{\Gamma}(\xi) = i(a^*(\xi) + a(\xi)) . \quad (4.7)
\]

One checks that

\[
\{ \Gamma(\xi), \Gamma(\eta) \} = \{ \bar{\Gamma}(\xi), \bar{\Gamma}(\eta) \} = -2g(\xi, \eta) , \quad (4.8)
\]

\[
\{ \Gamma(\xi), \bar{\Gamma}(\eta) \} = 0 , \quad (4.9)
\]

for arbitrary \( \xi \) and \( \eta \) in \( \Omega^1(\mathbb{M}) \). Thus \( \Gamma(\xi) \) and \( \bar{\Gamma}(\xi), \xi \in \Omega^1(\mathbb{M}) \), are anti-commuting sections of two isomorphic Clifford bundles, \( Cl(M) \), over \( M \).

An \( n \)-dimensional Riemannian manifold \( (\mathbb{M}, g) \) is a spin\(^c \) manifold if and only if \( \mathbb{M} \) is oriented and there exists a complex Hermitian vector bundle \( S \) of rank \( 2^{|k|} \) over \( \mathbb{M} \) (where \( |k| \) denotes the integer part of \( k \in \mathbb{R} \)) and a bundle homomorphism \( c: T^*\mathbb{M} \longrightarrow \text{End}(S) \) such that

\[
c(\xi) + c^*(\xi) = 0 \quad (4.10)
\]

\[
c^*(\xi) c(\xi) = g(\xi, \xi) , \quad (4.11)
\]

for arbitrary \( \xi \in \Omega^1(\mathbb{M}) \). The adjoint \( c^*(\xi) \) of \( c(\xi) \) is defined (pointwise) with respect to the Hermitian structure \( \langle \cdot, \cdot \rangle_S \) on \( S \). The Hermitian structure \( \langle \cdot, \cdot \rangle_S \) and the Riemannian volume form \( d\text{vol}_g \) determine a scalar product, \( \langle \cdot, \cdot \rangle_S \), on the space \( \Gamma(S) \) of sections of \( S \).

The completion of \( \Gamma(S) \) in the norm determined by the scalar product \( \langle \cdot, \cdot \rangle_S \) is a Hilbert space denoted by \( \mathcal{H}_e \), the Hilbert space of square-integrable Pauli-Dirac spinors on \( \mathbb{M} \). The homomorphism \( c \) extends uniquely to an irreducible \( \ast \)-representation of the Clifford algebra over \( T^*_x\mathbb{M} \) on the fibre \( S_x \) of \( S \) over \( x \), for all \( x \in \mathbb{M} \).

If \( M \) is an even-dimensional spin\(^c \) manifold then there is a section \( \sigma \neq 0 \) of the Clifford bundle generated by the operators \( c(\xi), \xi \in \Omega^1(\mathbb{M}) \), which anti-commutes with every \( c(\xi) \) and satisfies \( \sigma^2 = 1 \), (\( \sigma \) corresponds to the Riemannian volume form on \( \mathbb{M} \)), and there is an isomorphism

\[
i : \Omega^*_\mathbb{C}(M) \longrightarrow \Gamma(S) \otimes_\mathcal{A} \Gamma(S) , \quad (4.12)
\]

where \( \mathcal{A} = C(M) \), and where \( S \) is the “charge-conjugate” bundle associated to \( S \), obtained from \( S \) by complex conjugation of the transition functions of \( S \). The bundle \( S \) inherits a
natural Clifford action $\tilde{c} : T^*M \rightarrow \text{End}(\tilde{S})$ from the Clifford action $c$ on $S$, and the isomorphism $i$ is an intertwiner satisfying

$$i \circ \Gamma(\xi) = (1 \otimes c(\xi)) \circ i,$$  \hspace{1cm} (4.13)

$$i \circ \tilde{\Gamma}(\xi) = (\tilde{c}(\xi) \otimes \sigma) \circ i,$$  \hspace{1cm} (4.14)

for all $\xi \in \Omega^1(M)$. The element $\sigma$ is inserted on the r.s. of (4.14) to ensure that the Clifford actions $\Gamma$ and $\tilde{\Gamma}$ anti-commute, as required in (4.9).

If $M$ is an odd-dimensional spin$^c$ manifold then the Clifford algebra associated with a cotangent space $T^*_xM$ contains a central element, $\sigma$, corresponding to parity. There is then an isomorphism

$$i : \Omega^\bullet_c(M) \rightarrow \Gamma(\tilde{S}) \otimes A \Gamma(S) \otimes \mathbb{C}^2$$  \hspace{1cm} (4.15)

such that

$$i \circ \Gamma(\xi) = (1 \otimes c(\xi) \otimes \tau_3) \circ i,$$  \hspace{1cm} (4.16)

$$i \circ \tilde{\Gamma}(\xi) = (\tilde{c}(\xi) \otimes 1 \otimes \tau_1) \circ i,$$

where $\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

A connection $\nabla_S$ on $S$ is called a spin$^c$ connection iff it satisfies the “Leibniz rule”

$$\nabla^S_X(c(\eta)\psi) = c(\nabla_X\eta)\psi + c(\eta)\nabla^S_X\psi,$$  \hspace{1cm} (4.17)

for arbitrary vector fields $X$, one-forms $\eta$ and sections $\psi \in \Gamma(S)$, where $\nabla$ is a connection on $T^*M$. We say that $\nabla^S$ is compatible with the Levi-Civita connection iff, in (4.17), $\nabla = \nabla^{L.C.}$.

If $\nabla^S_1$ and $\nabla^S_2$ are two Hermitian spin$^c$ connections compatible with the same connection $\nabla$ on $T^*M$ then

$$(\nabla^S_1 - \nabla^S_2)\psi = i\alpha \otimes \psi$$  \hspace{1cm} (4.18)

for some real one-form $\alpha \in \Omega^1(M)$. The physical interpretation of $\alpha$ is that it is the difference of two electromagnetic vector potentials. If $R_{\nabla_S}$ denotes the curvature of a spin$^c$ connection $\nabla^S$ then

$$2^{-|S|/2} \text{tr} \left( R_{\nabla_S} (X,Y) \right) = F_A(X,Y),$$  \hspace{1cm} (4.19)

for arbitrary vector fields $X,Y$, where $F_{2A}$ is the curvature (“electromagnetic field strength”) of a U(1)-connection $2A$ (“vector potential”) on a line bundle canonically associated to $S$; $A$ itself defines a “virtual U(1)-connection”. See [20] and [18] for details.

The Pauli-Dirac operator associated with a spin$^c$ connection $\nabla^S$ is defined by

$$D_A = c \circ \nabla^S.$$  \hspace{1cm} (4.20)

We are now prepared to describe Pauli’s quantum theory of the non-relativistic electron. The Hilbert space of pure state vectors of a one-electron system is chosen to be $\mathcal{H}_e$, the space of square-integrable Pauli-Dirac spinors. The dynamics of an electron, with gyromagnetic factor $g$ measuring the strength of the magnetic moment of the electron set equal to 2, is given by the Hamiltonian

$$H_A = \frac{\hbar^2}{2m} D_A^2 + v = \frac{\hbar^2}{2m} \left( -\Delta^S_A + \frac{r}{4} + c(F_A) \right) + v,$$  \hspace{1cm} (4.21)
where \( m \) is the mass of the electron, \( v \) is the scalar ("electro-static") potential, which is a function on \( M \), \( r \) denotes the scalar curvature, \( \triangle^S_A \) is the “Lichnerowicz (covariant) Laplacian”, and \( c(F_A) \) denotes Clifford multiplication by the 2-form \( F_A \). For conditions ensuring that \( H_A \) is bounded from below and self-adjoint see [17,20,21].

Considering position measurements as fundamental, one chooses \( \mathcal{A} := C(M) \) as an algebra of observables. From the point of view of quantum physics it is, however, more natural to choose the algebra \( \mathcal{F} \) of “functions on quantum phase space” as an algebra of observables. The (non-commutative) algebra \( \mathcal{F} \) is defined to be the smallest \( C^* \)-algebra generated by \( \{ \alpha_\tau(a), f(H^0_A) \mid a \in C(M), \tau \in \mathbb{R}, f \in C_0^\infty(\mathbb{R}) \} \) where \( H^0_A = \hbar^2 2m D^2_A \) and

\[
\alpha_\tau(B) := e^{i\tau H^0_A/\hbar} B e^{-i\tau H^0_A/\hbar},
\]

for any \( B \in B(\mathcal{H}_e) \).

Connes has shown that the spectral triple \( (\mathcal{A}, \mathcal{H}_e, D_A) \) encodes the topology and Riemannian geometry of \( M \) completely; see [5]. It is less clear how much information about \( M \) is encoded into the data \( (\mathcal{F}, \mathcal{H}_e, D_A) \), viewed as abstract spectral data, and some interesting mathematical questions remain to be solved.

If \( v = 0 \) then

\[
H^0_A = D^2
\]

where \( D := \sqrt{\frac{\hbar}{2m}} D_A \), \( (D \text{ is self-adjoint}) \), i.e., \( H^0_A \) is the square of a “supercharge”. If \( M \) is even-dimensional then, as discussed above, the Clifford bundle over \( M \) has a section \( \sigma \) which is a unitary involution on \( \mathcal{H}_e \) with the property that

\[
[\sigma, a] = 0 \quad \text{for all} \quad a \in \mathcal{A} \quad (a \in \mathcal{F}),
\]

but

\[
\{\sigma, D\} = 0.
\]

Then \( \sigma \) defines a \( \mathbb{Z}_2 \)-grading of \( \mathcal{H}_e \). The data \( (\mathcal{A}, \mathcal{H}_e, D, \sigma) \) yield an example of \( N = 1 \) supersymmetric quantum mechanics\(^*\). An important topological invariant of \( M \) provided by Pauli’s supersymmetric quantum mechanics of a non-relativistic electron is given by the index of \( D \),

\[
\text{str} \left( e^{-\beta H^0_A} \right) := \text{tr} \left( \sigma e^{-\beta H^0_A} \right)
\]

which is easily seen to be independent of \( \beta \). Using a fairly standard Feynman-Kac formula to express (4.24) as a functional integral and studying the small \( \beta \) asymptotics of this integral, Alvarez-Gaumé has been able to rederive the \( \hat{A} \) genus and the index density for the Dirac operator \( D \) in a simple manner; see [22] (and [17]). We shall not pursue this theme here.

In order to describe the twin of Pauli’s electron, the non-relativistic positron, we replace the bundle \( S \) by the charge-conjugate spinor bundle \( \bar{S} \). A Hermitian spin connection \( \nabla^S \) on \( S \) uniquely determines a spin connection \( \nabla^\bar{S} \) on \( \bar{S} \), by setting \( \nabla^\bar{S}_X = C \nabla^S_X C^{-1} \), where \( C : S \longrightarrow \bar{S} \) is charge conjugation, and \( X \) is an arbitrary real

\(^*\)Our nomenclature deviates from the one used in the older literature, e.g. in [22]!
vector field. The space of square integrable sections of $\bar{S}$ is denoted by $\mathcal{H}_p$; $\mathcal{H}_p$ is a right module, while $\mathcal{H}_e$ is a left module for $\mathcal{A}$ and $\text{Cl}(M)$. One defines

$$\bar{D}_A = \bar{c} \circ \nabla \bar{S}$$

and sets

$$\bar{H}_A = \frac{\hbar^2}{2m} \bar{D}_A^2 - \nu.$$  \hspace{1cm} (4.25)

The physical interpretation of these changes is simply that the sign of the electric charge of the particle is reversed, replacing $A$ by $-A$ (and $\nu$ by $-\nu$), keeping everything else, such as its mass $m$, unchanged.

The third character of the play is the (non-relativistic) positronium, the particle corresponding to a ground state of a bound pair of an electron and a positron. As an algebra of “observables” we continue to use e.g. $\mathcal{A} = C(M)$, as for the electron and the positron.

The Hilbert space of pure state vectors of positronium is

$$\mathcal{H}_{e-p} = \mathcal{H}_p \otimes \mathcal{H}_e,$$  \hspace{1cm} (4.26)

and

$$\mathcal{H}_{e-p} = (\mathcal{H}_p \otimes \mathcal{A} \mathcal{H}_e)_+ \oplus (\mathcal{H}_p \otimes \mathcal{H}_e)_-$$

$$\cong (\mathcal{H}_p \otimes \mathcal{A} \mathcal{H}_e) \otimes \mathbb{C}^2,$$  \hspace{1cm} (4.27)

dim $M$ odd.

Elements in $(\mathcal{H}_p \otimes \mathcal{A} \mathcal{H}_e)_+$ are even, elements $(\mathcal{H}_p \otimes \mathcal{A} \mathcal{H}_e)_-$ are odd under space reflection. By (4.12) and (4.15), the Hilbert space $\mathcal{H}_{e-p}$ is the Hilbert space (4.1) of square-integrable differential forms on $M$. A connection $\tilde{\nabla}$ on $\mathcal{H}_{e-p}$ can be defined as follows: If $\phi \in \mathcal{H}_{e-p}$ is given by $\phi = \psi_1 \otimes \mathcal{A} \psi_2 (\otimes u)$, with $\psi_1 \in \mathcal{H}_p$, $\psi_2 \in \mathcal{H}_e$, $(u \in \mathbb{C}^2)$, we set

$$\tilde{\nabla} \phi = \left( \nabla^S \psi_1 \right) \otimes \mathcal{A} \psi_2 (\otimes u) + \psi_1 \otimes \mathcal{A} \nabla^S \psi_2 (\otimes u).$$  \hspace{1cm} (4.28)

If $\nabla^S$ is compatible with the connection $\nabla$ on $T^*M$ then $\tilde{\nabla}$ is the connection on $\Lambda^\bullet M$ determined by $\nabla$. We note that $\tilde{\nabla}$ is independent of the electromagnetic vector potential $A$ (the virtual U(1)-connection), which, physically, comes from the fact that the electric charge of positronium is zero.

We define two first-order differential operators on $\mathcal{H}_{e-p}$ by

$$\mathcal{D} = \Gamma \circ \tilde{\nabla}, \quad \bar{\mathcal{D}} = \bar{\Gamma} \circ \tilde{\nabla},$$  \hspace{1cm} (4.29)

with $\Gamma$ and $\bar{\Gamma}$ defined as in (4.13), (4.14), (4.16) (see also (4.6,7)). If $\nabla^S$ is compatible with the Levi-Civita connection $\nabla^{L.C.}$ then $\mathcal{D}$ and $\bar{\mathcal{D}}$ satisfy the algebra

$$\{ \mathcal{D}, \bar{\mathcal{D}} \} = 0, \quad \mathcal{D}^2 = \bar{\mathcal{D}}^2$$  \hspace{1cm} (4.30)

defining $N = (1,1)$ supersymmetry.

The quantum theory of non-relativistic positronium is formulated in terms of the $N = (1,1)$ spectral data

$$(\mathcal{A}, \mathcal{H}_{e-p}, \mathcal{D}, \bar{\mathcal{D}}).$$  \hspace{1cm} (4.31)
and its dynamics is determined by the Hamiltonian

\[
H = \frac{\hbar^2}{2\mu} D^2 = \frac{\hbar^2}{2\mu} \bar{D}^2 ,
\]

(4.32)

where \( \mu = 2m \) is the mass of positronium. Such data are meaningful even for manifolds that are not spin\(^c\); in physics jargon, one could say that, on manifolds which do not carry a spinor bundle \( S \), an electron and a positron are “permanently confined” to a positronium bound state.

The Weitzenböck formula says that

\[
H = \frac{\hbar^2}{2\mu} \left( -\Delta + \frac{r}{4} - \frac{1}{8} R_{ijkl} \bar{\Gamma}^{ij} \Gamma^{kl} \right) ,
\]

(4.33)

where \(-\Delta = \nabla^i g^{ij} \nabla_j = -g^{ij} (\nabla_i \nabla_j - \Gamma^k_{ij} \nabla_k)\) is the Bochner Laplacian, \( \Gamma^k_{ij} \) are the Christoffel symbols of the Levi-Civita connection, \( r \) is scalar curvature, and \( R_{ijkl} \) are the components of the Riemann curvature tensor, all in local coordinates \( q_j, j = 1, \ldots, n \), on \( M \); finally \( \Gamma^j = \Gamma(dq^j), \bar{\Gamma}^j = \bar{\Gamma}(dq^j) \), and the summation convention is used in (4.33). One recognizes the r.s. of (4.33) to be proportional to the Laplacian on the space of differential forms. This is not surprising: We introduce two operators \( \mathbb{d} \) and \( \mathbb{d}^* \) by

\[
\mathbb{d} = \frac{1}{2} (D - i\bar{D}) , \quad \mathbb{d}^* = \frac{1}{2} (D + i\bar{D}) .
\]

(4.34)

Then the relations (4.30) show that

\[
\mathbb{d}^2 = (\mathbb{d}^*)^2 = 0 , \quad H = \frac{\hbar^2}{2\mu} (\mathbb{d} \mathbb{d}^* + \mathbb{d}^* \mathbb{d}) .
\]

(4.35)

Using (4.6), (4.7) and (4.29), (4.34), one easily verifies that

\[
\mathbb{d} = a^* \circ \bar{\nabla} = a \circ \bar{\nabla} ,
\]

(4.36)

where \( a^* \) is defined in (4.3), and \( a \) denotes anti-symmetrization; in local coordinates, \( \mathbb{d} = a^*(dq^j)\bar{\nabla}_j \). Since the torsion \( T(\bar{\nabla}) \) of a connection \( \bar{\nabla} \) on \( \Omega^* (M) \) is defined by

\[
T(\bar{\nabla}) = d - a \circ \bar{\nabla} ,
\]

(4.37)

where \( d \) denotes exterior differentiation, we conclude that

\[
\mathbb{d} = d \iff T(\bar{\nabla}) = 0 \iff (4.30) \text{ holds} ,
\]

assuming that, in (4.30), \( D \) and \( \bar{D} \) are self-adjoint operators on \( \mathcal{H}_{e-p} \), which is implied, formally, by the assumption that \( \bar{\nabla} \) is a Hermitian connection. Thus \( \mathbb{d} = d \) is exterior differentiation precisely if \( \bar{\nabla} \) is the Levi-Civita connection on \( \mathcal{H}_{e-p} \).

It follows that the \( N = (1, 1) \) supersymmetric quantum theory of non-relativistic positronium can be formulated on general, orientable Riemannian manifolds \( (M, g) \) which need not be spin\(^c\).

If \( \gamma \) is the operator on \( \mathcal{H}_{e-p} \) with eigenvalue \(+1\) on forms of even degree and \(-1\) on forms of odd degree then

\[
\{ \gamma, \mathbb{d} \} = \{ \gamma, \mathbb{d}^* \} = 0 , \quad [\gamma, a] = 0 ,
\]

(4.38)
for all \( a \in A \).

An algebra \( F \) of “functions on quantum phase space” can be defined as in (2.8), (2.9): \( F \) is generated by

\[
\{ \alpha_\tau(a), f(H) \mid a \in A, \tau \in \mathbb{R}, f \in C_0^\infty(\mathbb{R}) \},
\]

with

\[
\alpha_\tau(A) = e^{i\tau H/\hbar} A e^{-i\tau H/\hbar},
\]

for all \( A \in B(\mathcal{H}_{e-p}), \tau \in \mathbb{R} \).

The spectral data \((A, \mathcal{H}_{e-p}, \mathcal{D}, \alpha, \gamma)\) or \((F, \mathcal{H}_{e-p}, \mathcal{D}, \alpha, \gamma)\) define an example of \( N = (1,1) \) supersymmetric quantum mechanics: There are two supercharges \( \mathcal{D} \) and \( \mathcal{D}^* \) (or \( D \) and \( \bar{D} \)) satisfying the algebra (4.35) (or (4.30)). When \( \mathcal{D} = d \) (exterior differentiation) the \( \mathbb{Z}_2 \)-grading \( \gamma \) can be replaced by a \( \mathbb{Z}_2 \)-grading \( T \) counting the degree of a differential form. Furthermore, if \( M \) is orientable one can define a unitary Hodge involution, \( * \), on \( \mathcal{H}_{e-p} \) such that \( *d^{*^{-1}} = \zeta d^*, |\zeta| = 1 \), and \( *a^{*^{-1}} = a \), for all \( a \in A \) (or \( a \in F \)). If \( M \) is even-dimensional then \( * \) can be constructed from the element \( \sigma \) anti-commuting with \( i \circ \Gamma(\xi) \circ i^{-1} \) and commuting with \( i \circ \bar{\Gamma}(\xi) \circ i^{-1} \), for all \( \xi \in \Omega^1(M) \):

\[
* = \sigma.
\]

\( N = (1,1) \) supersymmetric quantum theory yields topological invariants for \( M \) if \( M \) is even-dimensional:

the \textit{Euler number}

\[
\chi(M) = \text{tr} \left( \gamma e^{-\beta H} \right),
\]

and the \textit{Hirzebruch signature}

\[
\tau(M) = \text{tr} \left( * e^{-\beta H} \right).
\]

Since \( \gamma \) and \( * = \sigma \) anti-commute with \( D \), the r.s. of (4.41) and (4.42) are easily seen to be independent of \( \beta \) and of the metric on \( M \). Using a path integral representation of the r.s. of (4.41), one derives the Gauss-Bonnet formula. Similarly, (4.42) can be evaluated in terms of the Hirzebruch polynomial; see [22,17].

In Section 1, we have considered the equations of motion for a classical, relativistic scalar particle and have derived them from an action principle with an action \( S \) given in eq. (1.4). In this section, we study \textit{quantum-mechanical, non-relativistic} particles with \textit{spin}. Space-time is given by \( N := M \times \mathbb{R} \), where \( M \) is space, \( x^0 = \tau \in \mathbb{R} \), and

\[
g_{\mu\nu} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & -g_{ij}
\end{pmatrix},
\]

where \( g = (g_{ij}) \) is the Riemannian metric on \( M \). In this situation the action of a scalar particle with mass \( \mu \) is given by

\[
S \left( x(\cdot) \right) = -\frac{\mu}{2} \int g_{ij}(x(\tau)) \frac{dx^i(\tau)}{d\tau} \cdot \frac{dx^j(\tau)}{d\tau} \ d\tau,
\]

(4.43)
where, now, $\tau$ is time. Quantum-mechanically, the Hamiltonian of a scalar particle is given by

$$H = -\frac{\hbar^2}{2\mu} \triangle_g ,$$

where $\triangle_g$ is the Laplace-Beltrami operator acting on

$$\mathcal{H} = L^2(M, d\text{vol}_g) .$$

According to Feynman and Kac, the heat kernel $(e^{-\beta H/\hbar}(x,y), x,y \in M)$, is given by

$$(e^{-\beta H/\hbar})(x,y) = \int_{x(0)=x}^{x(\beta)=y} \prod_{\tau \in [0,\beta]} dx(\tau) ,$$

where $S_\beta$ is given by (4.43) with the $\tau$-integration extending over the interval $[0,\beta]$, and $dx(\tau) := d\text{vol}_g(x(\tau))$. The mathematical interpretation of the integrand on the r.s. of (4.44) is that it is given by the Wiener measure on path space $X \tau \in [0,\beta] M_{\tau}$, where $M_{\tau}$ is a copy of $M$ (with $M$ compact) and the Cartesian product is equipped with the Tychonov topology; see e.g. [23].

In order to evaluate expressions like (4.24), (4.41) or (4.42), we require a generalization of formulae (4.43,44) for particles with spin. People who know their path integral formulation of non-relativistic many-body theory will have little difficulty in finding such a generalization (see e.g. [22,24], and [25] for some details). As an example, we consider the Hamiltonian $H$ given in (4.33) and we propose to derive a path integral representation for the heat kernel corresponding to $H$. In order to be explicit, we work in a local coordinate patch of $M$, with coordinate functions now denoted by $x^1, \ldots, x^n$. It is advantageous to reexpress the r.s. of (4.33) in terms of the creation and annihilation operators $a^* := a^*(dx^j)$, $a^j := a(dx^j) \equiv g^{ij} a(\partial_i)$ defined in (4.3), (4.4), respectively. Then

$$H = \frac{\hbar^2}{2\mu} (-\triangle - R_{ijkl} a^*^i a^j a^*^k a^l)$$

with

$$-\triangle = \nabla^*_i g^{ij} \nabla_j , \quad \nabla_j = \partial_j - \Gamma^k_{jl} a^l a_k ,$$

where $a_k = g_{km} a^m$. As usual in the functional integral formulation of non-relativistic fermions (see e.g. [25]), we now associate Grassmann variables $\psi^j(\tau)$ with $a^j$, and Grassmann variables $\psi^j(\tau)$ with $a^j$, $\tau \in \mathbb{R}$, such that

$$\{\psi^i(\tau), \psi^j(\tau')\} = \{\psi^i(\tau), \psi^j(\tau')\} = \{\psi^i(\tau), \psi^j(\tau')\} = 0 .$$

The action corresponding to (4.45) is then given by

$$S_\beta (x, \psi, \psi^*) = -\mu \int_0^\beta \left[ \frac{1}{2} g_{jk}(x(\tau)) \frac{dx^j(\tau)}{d\tau} \cdot \frac{dx^k(\tau)}{d\tau} + ig_{jk}(x(\tau)) \psi^j(\tau) D_\tau \psi^k(\tau) - \frac{1}{2} \left( \frac{\hbar}{\mu} \right)^2 R_{ijkl}(x(\tau)) \psi^i(\tau) \psi^j(\tau) \psi^k(\tau) \psi^l(\tau) \right] d\tau ,$$

(4.47)
where
\[ D_\tau \psi^k(\tau) = \frac{d \psi^k(\tau)}{d\tau} + \Gamma^k_{lm} \frac{d \psi^l(\tau)}{d\tau} \psi^m(\tau). \]

Then
\[
\chi(M) = \text{tr} (\gamma e^{-\beta H}) = \int e^{\frac{1}{\hbar} S_B(x,\psi,\bar{\psi})} \prod_{\tau \in [0,\beta]} dx(\tau) \prod_{j=1}^n d \psi^j(\tau) d \bar{\psi}^j(\tau), \tag{4.48}
\]
and periodic boundary conditions are imposed at \( \tau = 0, \beta \). Hence, for very small \( \beta \), the constant modes dominate the functional integral (4.48). It is then easy to evaluate it (asymptotically, as \( \beta \to 0 \)) using the saddle point method. The result is the general Gauss-Bonnet formula. The calculations of \( \tau(M) \) and of (4.24) (index of \( D \)) are a little harder, although the basic ideas are the same; see [22], and [17] for rigorous proofs.

We do not want to enter into more detail, but rather continue our journey through non-relativistic quantum theory. Below (4.39), we have identified the spectral data
\[
(\mathcal{A}, \mathcal{H}, \mathbb{D}, \mathbb{D}^*, \gamma), \tag{4.49}
\]
with relations (4.35), (4.38) of \( N = (1,1) \) supersymmetric quantum theory of which non-relativistic positronium is an example if one takes \( \mathcal{A} = C(M), \mathcal{H} = \mathcal{H}_{e-p}, \mathbb{D}^# = d^# \) (\( x^# \) denotes \( x \) or \( x^* \)), and for \( \gamma \) the operator detecting the parity of a differential form. In this example, the data (4.49) completely encode the differential topology and Riemannian geometry of \((M, g)\). Furthermore, they can be completed to
\[
(\mathcal{A}, \mathcal{H}, \mathbb{D}, \mathbb{D}^*, T, *) \tag{4.50}
\]
where \( T \) counts the degree of a differential form, and \( * \) is the Hodge operator, with \( *\mathbb{D}^* = \zeta \mathbb{D}^*, \ |\zeta| = 1 \), and \( *a^{-1} = a \), for all \( a \in \mathcal{A} \). We say that the data (4.50) define some \( N = (1,1) \) supersymmetric quantum theory. It is important to distinguish \( N = (1,1) \) from \( N = (1,1) \) supersymmetry: Every \( N = (1,1) \) supersymmetry is an \( N = (1,1) \) supersymmetry, but it may turn out to be impossible to enlarge \( N = (1,1) \) to \( N = (1,1) \) supersymmetry, even in the context of quantum theory on classical manifolds \((M, g)\). An example is provided by choosing a connection \( \nabla \) on \( T^*M \) with non-vanishing torsion \( (T(\nabla) \neq 0) \). We assume that the torsion of \( \nabla \) defines a closed three-form, denoted \( \vartheta \). Locally, in some coordinate patch of \( M \), we can then construct a 2-form \( \beta \) with \( d\beta = \vartheta \) and define an operator \( B := \beta \wedge = \beta_{ij} a^i a^j \). We then define a new “exterior derivative”
\[
d_\lambda := e^{\lambda B} d e^{-\lambda B}.
\]
Clearly \( d_\lambda^2 = (d_\lambda)^2 = 0 \). Two Pauli–Dirac operators can now be defined by
\[
\mathcal{D} := d_\lambda + d_\lambda^* \quad \text{and} \quad \mathcal{D} := i (d_\lambda - d_\lambda^*),
\]
for arbitrary real \( \lambda \). Since \( d_\lambda^2 = (d_\lambda^*)^2 = 0 \), \( \mathcal{D} \) and \( \mathcal{D} \) obey the \( N = (1,1) \) algebra (see also [24] where a specific choice for \( \lambda \) is made). Of course, there is no natural \( \mathbb{Z} \)-grading operator \( T \) in this example. If the form \( \vartheta \) is not exact (i.e., there does not exist a globally defined 2-form \( \beta \) with \( d\beta = \vartheta \)) then \( d_\lambda \), for \( \lambda \neq 0 \), and \( d = d_{\lambda=0} \) may give rise to
different cohomologies; see [24] for examples and expressions for the action functionals corresponding to $D^2$. We shall return to these issues in Sects. 4.2 and 5.

Our findings can be summarized as follows: Pauli's quantum theory of a non-relativistic electron, such as described by the $N = 1$ spectral data $(A, \mathcal{H}_{e-p}, D_A)$, with $A = C(M)$, or of positronium, such as described by the $N = (1,1)$ spectral data $(A, \mathcal{H}_{e-p}, d, d^*, T, *)$, completely encode the topology and geometry of the Riemannian manifold $(M, g)$. When $A$ is replaced by an algebra $\mathcal{F}$ of “functions over quantum phase space”, there remain interesting mathematical problems to be reckoned with, which we plan to discuss in future work — see also Section 5.

Readers not interested in quantum physics may ask what one gains by reformulating differential topology and geometry in terms of spectral data, such as those provided by $N = 1$ (electron) or $N = (1,1)$ (positronium) supersymmetric quantum mechanics, beyond a slick algebraic reformulation. The answer — as emphasized by Connes — is generality! Supersymmetric quantum mechanics enables us to study highly singular spaces or discrete objects, like graphs, lattices and aperiodic tilings (see e.g. [5]), and also non-commutative spaces, like quantum groups, as geometric spaces, and to extend standard constructions and tools of algebraic topology or of differential geometry to this more general context, so as to yield non-trivial results. Moreover, as we have argued in Section 3, quantum physics ultimately forces us to generalize the basic notions and concepts of geometry.

The principle that the time evolution of a quantum mechanical system is a one-parameter unitary group on a Hilbert space, whose generator is the Hamiltonian of the system (a self-adjoint operator), entails that the study of supersymmetric quantum mechanics is the study of metric geometry. Let us ask then how we would study manifolds like symplectic manifolds that are, a priori, not endowed with a metric. The example of symplectic manifolds is instructive, so we sketch what one does (see [18]).

Let $(M, \omega)$ be a symplectic manifold. The symplectic form $\omega$ is a globally defined closed 2-form. It is known that every symplectic manifold can be equipped with an almost complex structure $J$ such that the tensor $g$ defined by

$$g(X,Y) = -\omega(JX,Y), \quad (4.51)$$

for all vector fields $X,Y$, is a Riemannian metric on $M$. Thus, we can study the Riemannian manifold $(M, g)$ with $g$ from (4.51) by exploring the quantum mechanical propagation of e.g. positronium on $M$, using the spectral data $(A, \mathcal{H}_{e-p}, d, d^*, T, *)$ of $N = (1,1)$ supersymmetric quantum mechanics, with $A = C(M)$. We must ask how these data “know” that $M$ is symplectic. The answer is as follows: We can view the $\mathbb{Z}$-grading $T$ as the generator of a U(1)-symmetry (a “global gauge symmetry”) of the system. It may happen that this symmetry can be enlarged to an SU(2)-symmetry, with generators $L^1, L^2, L^3$ acting on $\mathcal{H}_{e-p}$ such that they commute with all elements of $A$ and have the following additional properties:

- $L^3 = T - \frac{n}{2}$ with $n = \text{dim} M$.

Defining $L^\pm = L^1 \pm i L^2$, the structure equations of su(2) = Lie(SU(2)) imply that

- $[L^3, L^\pm] = \pm 2 L^\pm$, $[L^+, L^-] = L^3$,

and, since in quantum mechanics symmetries are represented unitarily,

- $(L^3)^* = L^3$, $(L^\pm)^* = L^\mp$.

We also assume that
iv) \([L^+,d] = 0\), hence \(L^-\) commutes with \(d^*\) by property iii). Next we define an operator \(\tilde{d}^*\) by

\[
\tilde{d}^* = [L^-,d] ;
\]

it satisfies \([L^+,\tilde{d}^*] = d\) because of ii) and iii), and also

\[
\{\tilde{d}^*,d\} = 0
\]

since \(d\) is nilpotent. Assuming, moreover, that

v) \([L^-,\tilde{d}^*] = 0\)

we find that \((d,\tilde{d}^*)\) transforms as a doublet under the adjoint action of \(L^3, L^+, L^-\) and that \(\tilde{d}^*\) is nilpotent. Thus, \((\tilde{d},-d^*)\) with \(\tilde{d} = (\tilde{d}^*)^*\) is an SU(2)-doublet, too, and \(\tilde{d}^2 = 0\).

The theorem is that the spectral data

\[
(\mathcal{A}, \mathcal{H}_{\text{e-p}}, d, d^*, \{L^3, L^+, L^-, d, d^*\}, *) ,
\]

with properties i) - v) assumed to be valid, encode the geometry of a symplectic manifold \((M,\omega)\) equipped with the metric \(g\) defined in (4.51). The identifications are as follows:

\[
L^3 = T - \frac{n}{2} , \quad L^+ = \omega \wedge = \frac{1}{2} \omega_{ij} a^i a^* j ,
\]

\[
L^- = (L^+)^* = \frac{1}{2} (\omega^{-1})_{ij} a_i a_j .
\]

Assumption iv) is equivalent to \(d\omega = 0\). Further details can be found in [18].

We say that the spectral data (4.53) define \(N = 4^s\) supersymmetric quantum mechanics, because there are four “supersymmetry generators” \(d, \tilde{d}^*, \tilde{d}, d^*\); the superscript \(s\) stands for “symplectic”.

Note that we are not claiming that

\[
\{d, \tilde{d}\} = 0
\]

because this equation does, in general, not hold. However, if it holds then \((M,\omega)\) is in fact a Kähler manifold, with the \(J\) from eq. (4.51) as its complex structure and \(\omega\) as its Kähler form. Defining

\[
\partial = \frac{1}{2} (d - i\tilde{d}) , \quad \bar{\partial} = \frac{1}{2} (d + i\tilde{d}) ,
\]

one finds that, thanks to eqs. (4.52, 4.54) and because \(d\) and \(\tilde{d}\) are nilpotent,

\[
\partial^2 = \bar{\partial}^2 = 0 , \quad \{\partial,\partial^*\} = \{\bar{\partial},\bar{\partial}^*\} .
\]

There is a useful alternative way of saying what it is that identifies a symplectic manifold \((M,\omega)\) as a Kähler manifold: Eq. (4.54) is a consequence of the assumption that an \(N = 4^s\) supersymmetric quantum mechanical model has an additional U(1)-symmetry — which, in physics jargon, one is tempted to call a “global chiral U(1)-gauge symmetry”:

We define

\[
d_\theta = \cos \theta \, d + \sin \theta \, \tilde{d} ,
\]

\[
\tilde{d}_\theta = -\sin \theta \, d + \cos \theta \, \tilde{d} ,
\]

\[
(4.57)
\]
and assume that \((d_\theta, \tilde{d}_\theta)\) and \((\tilde{d}_\theta, -d^*_\theta)\) are again SU(2)-doublets with the same properties as \((d, d^*)\) and \((\tilde{d}, -d^*_\theta)\), for all real angles \(\theta\). Then the nilpotency of \(d, \tilde{d}\) and of \(\tilde{d}_\theta\) for all \(\theta\) implies eq. (4.54). Furthermore

\[
\partial_\theta = \frac{1}{2} (d_\theta - i\tilde{d}_\theta) = e^{i\theta} \partial, \quad \bar{\partial}_\theta = \frac{1}{2} (d_\theta + i\tilde{d}_\theta) = e^{-i\theta} \bar{\partial}.
\]

Assuming that the symmetry (4.57) is implemented by a one-parameter unitary group on \(H_{e-p}\) with an infinitesimal generator denoted by \(J_0\), we find that

\[
[J_0, d] = -i \tilde{d}, \quad [J_0, \tilde{d}] = id.
\]  

Geometrically, \(J_0\) can be expressed in terms of the complex structure \(J\) on a Kähler manifold — it is bilinear in \(a^*\) and \(a\) with coefficients given by \(J\). Defining

\[
\mathcal{T} := \frac{1}{2} (L^3 + J_0), \quad \overline{\mathcal{T}} := \frac{1}{2} (L^3 - J_0),
\]  

one checks that

\[
[T, \partial] = \partial, \quad [\mathcal{T}, \partial] = 0, \\
[\overline{\mathcal{T}}, \partial] = 0, \quad [\overline{\mathcal{T}}, \bar{\partial}] = \bar{\partial}.
\]  

Thus \(\mathcal{T}\) is the holomorphic and \(\overline{\mathcal{T}}\) the anti-holomorphic \(\mathbb{Z}\)-grading of complex differential forms. The spectral data

\[
(A, H_{e-p}, d, d^*, \{L^3, L^+, L^-\}, J_0, *)
\]  

belong to \(N = 4^+\) supersymmetric quantum mechanics. We have seen that they contain the spectral data

\[
(A, H_{e-p}, \partial, \partial^*, \bar{\partial}, \bar{\partial}^*, \mathcal{T}, \overline{\mathcal{T}}, *)
\]  

characterizing Kähler manifolds. We say that these define \(N = (2, 2)\) supersymmetric quantum mechanics. If one drops the requirement that \(\partial\) anti-commutes with \(\bar{\partial}^*\) (amounting to the breaking of the SU(2) symmetry generated by \(L^3, L^+, L^-\)) the data (4.62) characterize complex Hermitian manifolds, see [18].

Alternatively, complex Hermitian manifolds can be described by \(N = (1, 1)\) spectral data, as in eq. (4.50), with an additional U(1) symmetry generated by a self-adjoint operator \(J_0\) with the property that \(d := i[J_0, d]\) is nilpotent, and different from \(d\). Then \(\tilde{d}\) and \(d\) anti-commute, and one may define \(\partial\) and \(\bar{\partial}\) through eqs. (4.55). One verifies that

\[
\partial^2 = \bar{\partial}^2 = 0 \quad \text{and} \quad \{\partial, \bar{\partial}\} = 0.
\]

Having proceeded thus far, one might think that on certain Kähler manifolds with special properties the U(1) symmetries generated by \(\mathcal{T}\) and \(\overline{\mathcal{T}}\) are embedded into SU(2) symmetries with generators \(T^3 = \mathcal{T}, T^+, T^-\) (analogously for the anti-holomorphic generators) which satisfy properties i) through v) from above, with \(d\) and \(d^*\) replaced by \(\partial\) and \(\partial^*\), and such that \(\tilde{\partial}^* = [\mathcal{T}^-, \partial]\) — as well as analogous relations for the anti-holomorphic generators. Alternatively, one might assume that, besides the SU(2) symmetry generated by \(L^3, L^+, L^-\) there are actually two “chiral” U(1)-symmetries with generators \(I_0\) and \(J_0\), enlarging the original U(1) symmetry.
Indeed, this kind of symmetry enhancement can happen, and what one finds are spectral data characterizing \textit{Hyperkähler manifolds}. The two ways of enlarging the SU(2)×U(1) symmetry of Kähler manifolds to larger symmetry groups characteristic of Hyperkähler manifolds are \textit{equivalent} by a theorem of Beauville; see e.g. [26]. The resulting spectral data define what is called \( N = (4,4) \) supersymmetric quantum mechanics, having two sets of four supercharges, \( \{ \partial, \bar{\partial}^*, \bar{\partial}^*, \bar{\partial} \} \) and \( \{ \tilde{\partial}, \tilde{\partial}^*, \tilde{\partial}^*, \tilde{\partial} \} \), with the property that each set transforms in the fundamental representation of Sp(4) — see [18] for the details. This yields the data of \( N = 8 \) supersymmetric quantum mechanics — from which we can climb on to \( N = (8,8) \) or \( N = 16 \) supersymmetric quantum mechanics and enter the realm of very rigid geometries of symmetric spaces with special holonomy groups [27,26].

Of course, the operators
\begin{align}
I &:= \exp(-i\pi I_0), & J &:= \exp(-i\pi J_0), & K &:= IJ
\end{align}

in the group of “chiral symmetries” of the spectral data of \( N = (4,4) \) supersymmetric quantum mechanics correspond to the three complex structures of Hyperkähler geometry. One may then try to go ahead and enlarge these “chiral” symmetries by adding further complex structures. This leads to the study of hypercomplex manifolds with many complex structures. See e.g. [28] and references therein for some formal considerations in this direction, and also Section 5.

We could now do our journey through the land of geometry and supersymmetric quantum mechanics in reverse and pass from special (rigid) geometries, i.e., supersymmetric quantum mechanics with high symmetry, to more general ones by reducing the supersymmetry algebra. The passage from special to more general geometries then appears in the form of \textit{supersymmetry breaking} in supersymmetric quantum mechanics (in a way that is apparent from our previous discussion). The symmetry generators in the formulation of geometry as supersymmetric quantum mechanics are bilinear expressions in the creation and annihilation operators \( a^* \) and \( a \) from eqs. (4.3,4) with coefficients that are tensors of rank two. It is quite straightforward to find conditions that guarantee that such tensors generate symmetries and hence to understand what kind of \textit{deformations} of geometry \textit{preserve} or \textit{break} the symmetries. Furthermore, the general transformation theory of quantum mechanics enables us to describe the deformation theory of the supersymmetry generators (\( D_A, \bar{D}, \bar{D}; \) or \( d, d^* \)) including isospectral deformations (as unitary transformations). Deformations of \( d \) and \( d^* \) played an important role in Witten’s proof of the Morse inequalities [16] and in exploring geometries involving anti-symmetric tensor fields such as torsion — recall the example described above — which are important in conformal field theory.

We hope we have made our main point clear: \textit{Pauli’s quantum mechanics of the non-relativistic electron and of positronium on a general manifold (along with its internal symmetries) neatly encodes and classifies all types of differential geometry}. See [18] for details.
4.2 The special case where $M$ is a Lie group

What is special if physical space $M$ is a Lie group $G$? We briefly discuss this special case, because it will help us to understand conformal field theory and the operator formalism for BRST cohomology. For simplicity, we only consider finite-dimensional, compact, connected, semi-simple Lie groups. The group is denoted by $G$, and $\mathfrak{g}$ denotes its Lie algebra. For each $g \in G$, we denoted by $L_g$ the left action of $g$ on $G$. The tangent maps of $L_g$ are denoted by $D_g$. The Lie algebra $\mathfrak{g}$ of $G$ can be viewed as the space of left-invariant vector fields: Let $\varphi$ be an arbitrary function on $G$ and let $X \in \Gamma(TG)$ be a vector field on $G$. Then $X$ is left-invariant if

$$D_g X(\varphi)(h) = X(\varphi)(g^{-1}h) \quad (4.64)$$

for all $h$ and $g$ in $G$. The space of left-invariant vector fields is canonically isomorphic to the tangent space $T_eG = \mathfrak{g}$ at the unit element $e \in G$. The Lie algebra $\mathfrak{g}$ acts on itself by

$$ad_X(Y) = [X,Y], \quad X,Y \in \mathfrak{g}; \quad (4.65)$$

(adjoint representation). We define a symmetric, $\mathfrak{g}$-invariant Killing form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g} \times \mathfrak{g}$ by

$$\langle X,Y \rangle := \text{tr} (ad_X \cdot ad_Y), \quad X,Y \in \mathfrak{g}, \quad (4.66)$$

which is non-degenerate (if $G$ is semi-simple) and negative-definite. The Killing form defines a Riemannian metric $g$ on $TG$ by setting

$$g(X,Y) := - \langle D_{h^{-1}}X,D_{h^{-1}}Y \rangle, \quad (4.67)$$

for arbitrary $X$ and $Y$ in $T_h G$.

The Haar measure $dg$ on $G$ corresponds to the volume form associated with (4.67), normalized such that $\int_G dg = 1$. For compact Lie groups $dg$ is invariant under the left action $L_h$ and under the right action $R_h$ of $h \in G$ on $G$. Corresponding to the right action $R$ of $G$ on $G$, one can define right-invariant vector fields on $G$. The left (or right) action of $G$ on $G$ can be used to show that the tangent bundle $TG$ is parallelizable and it determines a flat connection $\nabla_L$ (or $\nabla_R$, respectively) with non-vanishing torsion. By $\nabla$ we denote the Levi-Civita connection corresponding to the metric (4.67).

Obviously we have all the data necessary to define a model of $N = 1$ (electron) or $N = (1,1)$ (positronium) supersymmetric quantum theory of particle motion on a compact, connected, semi-simple Lie group $G$.

Let $\{T_i\}_{i=1}^n$, where $n = \text{dim } G$, be a basis of left-invariant vector fields on $G$, and let $\{\vartheta^i\}_{i=1}^n$ be the dual basis of 1-forms. The structure constants $f^k_{ij}$ of $g$ in the basis $\{T_i\}_{i=1}^n$ are defined by

$$[T_i,T_j] = f^k_{ij} T_k. \quad (4.68)$$

The coefficients of the metric $g$ in (4.67) in this basis are given by

$$g_{ij} = g(T_i,T_j) = - f^l_{ik} f^l_{jk}. \quad (4.69)$$

Using the $G$-invariance of the Killing form,

$$\langle [X,Y],Z \rangle + \langle Y,[X,Z] \rangle = 0,$$
for $X, Y, Z \in g$, one shows that metricity and vanishing torsion yield the following formula for the Levi-Civita connection on $TG$:

$$\nabla T_i = \frac{1}{2} f^k_{ij} \vartheta^k \otimes T_j \ .$$

(4.70)

As in (4.3) and (4.4), we define creation- and annihilation operators $a^*$ and $a$ by setting

$$c^j \equiv a^{*j} := \vartheta^j \wedge, \ b_j \equiv a_j := T_j \rightarrow \ .$$

(4.71)

They satisfy the canonical anti-commutation relations

$$\{c^i, c^j\} = \{b_i, b_j\} = 0, \ \{c^i, b_j\} = \delta^i_j \ .$$

(4.72)

Then the Levi-Civita connection on $T^*G$ is given by

$$\nabla = \vartheta^j \otimes \left(T_j - \frac{1}{2} f^k_{ij} c^i b_k\right) \ ,$$

(4.73)

where $T_j(\varphi)$ is the directional derivative of a function $\varphi$ on $G$ in the direction of $T_j$. Since the torsion of $\nabla$ vanishes, exterior differentiation on $G$ is given by

$$d = a^* \circ \nabla = c^j T_j - \frac{1}{2} f^k_{ij} c^i c^j b_k \ ,$$

(4.74)

with $d^2 = 0$. To physicists $d$ is known under the name of BRST charge (see e.g. [29]).

Since, by definition,

$$\nabla L T_i = 0, \ \nabla L \vartheta^i = 0$$

we find that

$$T(\nabla_L) = d - a^* \circ \nabla = d - a^* \circ \nabla + a^* \circ (\nabla - \nabla_L) = a^* \circ (\nabla - \nabla_L) = -\frac{1}{2} f^k_{ij} c^i c^j b_k \ .$$

(4.75)

The corresponding 3-form

$$\theta = -\frac{1}{2} f^k_{ij} c^i c^j = \frac{4}{3} \, \text{tr} \, (g^{-1}dg)^{\wedge 3}$$

is closed (the Jacobi identity implies $d\theta = 0$) but not exact. Locally, we can choose a 2-form $\beta$ with $d\beta = \theta$. Setting $B := \beta \wedge$, we can consider the deformed exterior derivatives

$$d_\lambda := e^{\lambda B} \, d \, e^{-\lambda B} \quad \text{and define} \quad \mathcal{D} := d_\lambda + d_\lambda^* \quad \text{and} \quad \mathcal{D}^* := i(d_\lambda - d_\lambda^*) \ .$$

Since $d_\lambda^2 = (d_\lambda)^2 = 0$, it follows that $\{\mathcal{D}, \mathcal{D}^*\} = 0$ and $\mathcal{D}_2 = \mathcal{D}^2$. For a suitable choice of $\lambda$ (see [24]) one finds that

$$\mathcal{D} = \Gamma^i \left(T_i - \frac{1}{12} f^k_{ij} \Gamma^j \Gamma^k\right) \ ,$$

$$\mathcal{D}^* = \tilde{\Gamma}^i \left(\tilde{T}_i - \frac{i}{12} f^k_{ij} \tilde{\Gamma}^j \tilde{\Gamma}^k\right) \ ,$$

(4.76)

where $\tilde{T}_i$ is the directional derivative defined by a right-invariant vector field, also denoted by $\tilde{T}_i$, $i = 1, \ldots, n$, and $\tilde{\vartheta}^1, \ldots, \tilde{\vartheta}^n$ is the dual basis of 1-forms. Moreover $\Gamma^i = \Gamma(\vartheta^i)$, $\tilde{\Gamma}^i = \Gamma(\tilde{\vartheta}^i)$. One finds that

$$\mathcal{D}^2 = \mathcal{D}^2 = g^\gamma T_i T_j + \frac{g^\gamma n}{24} \geq \frac{g^\gamma n}{24} \ ,$$

(4.77)
where $g^\vee$ is the dual Coxeter number of $G$. The lower bound (4.77) proves that the Hilbert space $\mathcal{H}_{e-p} = L^2(\Lambda^* G, d\text{vol}_g)$ does not contain any $d_\lambda$-closed vectors that are not $d_\lambda$-exact; the physicists call this phenomenon \textit{spontaneously broken supersymmetry}. It does, however, contain $d$-closed vectors that are not $d$-exact. This proves that $\vartheta$ is \textit{not} exact, hence $H^3(G) \neq 0$, while $H^2(G) = H^4(G) = 0$ under our hypotheses on $G$ (see e.g. [30]).

We proceed towards reviewing some standard facts of representation theory like the Peter-Weyl theorem etc., the reason being that notions very similar to those encountered in group representation theory will appear again in the study of two-dimensional conformal field theory!

Let $G$ be as above. By $I$ we denote the trivial representation and by $\mathcal{R} = \{1, I, J, \ldots\}$ the complete list of irreducible representations of $G$. Since $G$ has been assumed to be compact, its irreducible representations are all finite-dimensional and unitarizable. Given $I \in \mathcal{R}$, let $I^\vee$ denote the representation contragredient to $I$, which can be defined by the property that the tensor product representation $I \otimes I^\vee$ contains the trivial representation precisely once.

Besides the Hilbert spaces $\mathcal{H}_e$ and $\mathcal{H}_{e-p}$, we define the Hilbert space

\[ \mathcal{H} = L^2(G, dg) . \]  

(4.78)

This space is a bi-module for the group algebra $\mathbb{C}[G]$: It carries the left-regular and the right-regular representation which commute with each other. The dense subspace $S \subset \mathcal{H}$ of smooth functions on $G$ carries the left-regular representation of left-invariant vector fields and the right-regular representation of right-invariant vector fields, which commute with each other. The Peter–Weyl theorem says that

\[ \mathcal{H} = \bigoplus_{I \in \mathcal{R}} W_I \otimes W_I^\vee , \]

(4.79)

where $W_I$ is the representation space for the representation $I \in \mathcal{R}$, which carries a canonical scalar product with respect to which $I$ is unitary. Choosing orthonormal bases in the spaces $W_I$, the Peter–Weyl theorem can be formulated as saying that the matrix elements $I_{ij}(g), I \in \mathcal{R}, g \in G,$ of irreducible representations form an orthonormal basis of $\mathcal{H}$. Since $G$ is compact, every representation of $G$ is a direct sum of irreducible representations. Given $I$ and $J$ in $\mathcal{R}$, we can form the tensor product representation $I \otimes J$ and consider its decomposition into irreducible representations (Clebsch–Gordan series)

\[ I \otimes J = \bigoplus_{K \in \mathcal{R}} N^K_{IJ} K , \]

(4.80)

where $N^K_{IJ}$ is the multiplicity of $K$ in $I \otimes J$, which the physicists call \textit{fusion rule}. The fusion rule $N^K_{IJ}$ is equal to the number of distinct intertwiners

\[ V_\alpha(I, J|K) : W_K \longrightarrow W_I \otimes W_J , \]

(4.81)

$\alpha = 1, \ldots, N^K_{IJ}$, which are called \textit{Clebsch–Gordan matrices}. These matrices are isometries satisfying

\[ V_\alpha^*(I, J|K) \quad V_\beta^*(I, J|K) = \delta_{\alpha\beta} \quad V_\alpha(I, J|K) \quad V_\beta(I, J|K) = \delta_{\alpha\beta} \quad P_{W_K} , \]

where $V_\alpha^*$ is the dual Coxeter number of $G$. The lower bound (4.77) proves that the Hilbert space $\mathcal{H}_{e-p} = L^2(\Lambda^* G, d\text{vol}_g)$ does not contain any $d_\lambda$-closed vectors that are not $d_\lambda$-exact; the physicists call this phenomenon \textit{spontaneously broken supersymmetry}. It does, however, contain $d$-closed vectors that are not $d$-exact. This proves that $\vartheta$ is \textit{not} exact, hence $H^3(G) \neq 0$, while $H^2(G) = H^4(G) = 0$ under our hypotheses on $G$ (see e.g. [30]).

We proceed towards reviewing some standard facts of representation theory like the Peter-Weyl theorem etc., the reason being that notions very similar to those encountered in group representation theory will appear again in the study of two-dimensional conformal field theory!

Let $G$ be as above. By $I$ we denote the trivial representation and by $\mathcal{R} = \{1, I, J, \ldots\}$ the complete list of irreducible representations of $G$. Since $G$ has been assumed to be compact, its irreducible representations are all finite-dimensional and unitarizable. Given $I \in \mathcal{R}$, let $I^\vee$ denote the representation contragredient to $I$, which can be defined by the property that the tensor product representation $I \otimes I^\vee$ contains the trivial representation precisely once.

Besides the Hilbert spaces $\mathcal{H}_e$ and $\mathcal{H}_{e-p}$, we define the Hilbert space

\[ \mathcal{H} = L^2(G, dg) . \]  

(4.78)

This space is a bi-module for the group algebra $\mathbb{C}[G]$: It carries the left-regular and the right-regular representation which commute with each other. The dense subspace $S \subset \mathcal{H}$ of smooth functions on $G$ carries the left-regular representation of left-invariant vector fields and the right-regular representation of right-invariant vector fields, which commute with each other. The Peter–Weyl theorem says that

\[ \mathcal{H} = \bigoplus_{I \in \mathcal{R}} W_I \otimes W_I^\vee , \]

(4.79)

where $W_I$ is the representation space for the representation $I \in \mathcal{R}$, which carries a canonical scalar product with respect to which $I$ is unitary. Choosing orthonormal bases in the spaces $W_I$, the Peter–Weyl theorem can be formulated as saying that the matrix elements $I_{ij}(g), I \in \mathcal{R}, g \in G,$ of irreducible representations form an orthonormal basis of $\mathcal{H}$. Since $G$ is compact, every representation of $G$ is a direct sum of irreducible representations. Given $I$ and $J$ in $\mathcal{R}$, we can form the tensor product representation $I \otimes J$ and consider its decomposition into irreducible representations (Clebsch–Gordan series)

\[ I \otimes J = \bigoplus_{K \in \mathcal{R}} N^K_{IJ} K , \]

(4.80)

where $N^K_{IJ}$ is the multiplicity of $K$ in $I \otimes J$, which the physicists call \textit{fusion rule}. The fusion rule $N^K_{IJ}$ is equal to the number of distinct intertwiners

\[ V_\alpha(I, J|K) : W_K \longrightarrow W_I \otimes W_J , \]

(4.81)

$\alpha = 1, \ldots, N^K_{IJ}$, which are called \textit{Clebsch–Gordan matrices}. These matrices are isometries satisfying

\[ V_\alpha^*(I, J|K) \quad V_\beta^*(I, J|K) = \delta_{\alpha\beta} \quad V_\alpha(I, J|K) \quad V_\beta(I, J|K) = \delta_{\alpha\beta} \quad P_{W_K} , \]
where $P_{\alpha K}^{\alpha'}$ is the orthogonal projection onto the $\alpha^{th}$ copy, $W_{\alpha K}^{\alpha}$, of $W_K$ appearing in $W_{\alpha} \otimes W_{\alpha'}$.

We define

$$C_{ikn}(I, J|K)_{jlm} := \int_G I_{ij}(g) \, J_{kl}(g) \, K_{nn}(g) \, dg,$$  \hfill (4.82)

where $K_{mn}(g) = (K(g)^*)_{nm} = K_{nm}(g^{-1})$, because all irreducible representations are unitary. Then, using the left- and right-invariance of the Haar measure ($dg = d(hg) = d(gh)$ for any $h \in G$), one verifies immediately that

$$\sum_{i',k'} I_{i'i'}(h) \, J_{k'k}(h) \, C_{i'k'n}(I, J|K)_{jlm} = \sum_{m'} C_{ikn}(I, J|K)_{jlm} \, K(h)_{mm'},$$

and

$$\sum_{j',l'} C_{ikn}(I, J|K)_{j'l'm} \, I_{j'j}(h) \, J_{l'l}(h) = \sum_{n'} K(h)_{nn'} \, C_{ikn'}(I, J|K)_{jlm}.$$

One concludes without difficulty that

$$C(I, J|K) = \sum_{\alpha} V_{\alpha}(I, J|K) \otimes V_{\alpha}^*(I, J|K).$$  \hfill (4.83)

It follows from the definition of the constants $C_{ikn}(I, J|K)_{jlm}$ that they are the structure constants of the abelian $C^\ast$-algebra $A = C(G)$. Choosing the functions $I_{ij}()$, $I \in R$, as generators of $A$, we conclude from (4.82) that

$$I_{ij}(g) \, J_{kl}(g) = \sum_{K,m,n} C_{ikn}(I, J|K)_{jlm} \, K_{mn}(g).$$  \hfill (4.84)

Put differently, $C_{ikn}(I, J|K)_{jlm}$ is the matrix element of the operator $J_{kl}(\cdot) \in A$ between the vectors $I_{ij}(\cdot) \in H$ and $K_{mn}(\cdot) \in H$.

The group $G$ has a left- and a right-representation on the space $H_{e-p}$ of square-integrable differential forms over $G$: If $L_g$ (resp. $R_g$) is the diffeomorphism of $G$ determined by left (resp. right) multiplication by $g$ and $\alpha$ is a differential form on $G$ then

$$\lambda(g)\alpha := L_g^*\alpha, \quad \rho(g)\alpha := R_g^*\alpha,$$  \hfill (4.85)

where $\varphi^*\alpha$ denotes the pull back of $\alpha$ under the diffeomorphism $\varphi$, define the left (resp. right) representation of $G$ on $H_{e-p}$. Let $H_{e-p}^l$ denote the subspace of differential forms with the property that $\lambda_{|H_{e-p}^l} \cong I$, where $I$ is some representation of $G$, not necessarily irreducible. Then, for an arbitrary $\alpha \in H_{e-p}^l$,

$$d\alpha = Q_I \alpha,$$

where

$$Q_I = c^j i(T_j) - \frac{1}{2} f_{mn}^k c^m c^a b_k,$$  \hfill (4.86)

and $i = dI$ is the representation of the Lie algebra $\mathfrak{g}$ of $G$ corresponding to $I$. The “BRST operator” $Q_I$ is nilpotent:

$$Q_I^2 = 0$$  \hfill (4.87)
If $T$ is defined by

$$T = \sum_j c^j b_j$$  \hspace{1cm} (4.88)

then the eigenspace of $T$ corresponding to an eigenvalue $p = 0, 1, 2, \ldots, \dim G$ consists precisely of all square-integrable $p$-forms on $G$. Physicists call the grading operator $T$ the “ghost number operator”. The $k^{th}$ “BRST cohomology group” of $Q_I$ is given by

$$H^k_I = \ker Q_I \bigg|_{\mathcal{C}^k_I} / \im Q_I \bigg|_{\mathcal{C}^{k-1}_I} = H^k(g, dI),$$  \hspace{1cm} (4.89)

where $\mathcal{C}^k_I$ is the eigenspace of $T |_{\mathcal{H}^k_{e-p}}$ corresponding to the eigenvalue $k$. It is easy to check that $H^0_I$ consists of all functions (0-forms) in $\mathcal{H}^l_{e-p}$ invariant under $L_g, g \in G$.

It is known that the notions introduced here are meaningful under much weaker assumptions on $G$ (or $g$).

We have reviewed all these “elementary” notions and results in Lie group theory in more detail than may be bearable, because analogous notions and results will turn up in conformal field theory.

### 4.3 Supersymmetric quantum theory and geometry put into perspective

What we call $N = 1$ supersymmetric quantum mechanics (Pauli’s electron) is a structure that has played and important role in Connes’ exploration of the “metric aspect” of non-commutative geometry [5]. For the formulation of non-commutative geometry, it is a useful exercise to translate our discussion of the passage from $N = (1, 1)$ to $N = (2, 2)$ to $N = (4, 4)$ etc. supersymmetric quantum mechanics, accomplished by adding symmetries, into the language of $N = 1$ supersymmetric quantum mechanics, encoding spin$^c$–geometry enriched by additional symmetries.

In a more rudimentary, but concrete form, the $N = 1$ spectral data have been studied for a long time, by physicists and mathematicians alike. The work of Lichnerowicz on the Dirac operator (e.g. formula (4.21), and also the realization that for a compact spin manifold $M$ without boundary $r > 0$ implies that the index of $D = D_{A=0}$ vanishes) and also index theory (see [17] and refs. given there) can be viewed as the study of Pauli’s electron on a general manifold and hence of $N = 1$ supersymmetric quantum mechanics. In particular, the study of zero modes of $D_{A}$ has turned out to be important in topology.

Zero modes of $D_{A}$ play an important role in a concrete physics context as well. Electrons are fermions. The Hilbert space of pure state vectors of a system describing $N$ non-relativistic electrons which move on a manifold $M$ (identified with physical space) is thus given by

$$\mathcal{H}^{(N)} = \mathcal{H}^{eN}_e,$$  \hspace{1cm} (4.90)

where $\mathcal{H}_e$ is the one-electron Hilbert space introduced above after eq. (4.11).

Let us suppose that $K$ static nuclei of atomic numbers $Z_1, \ldots, Z_K$, with $Z_j \leq Z_e < \infty$ for all $j = 1, \ldots, K$, are present at the points $y_1, \ldots, y_K$ of physical space $M$. We also fix an arbitrary (virtual) $\text{U}(1)$–connection $A$ with $\text{div} A = 0$ ($A$ is the electromagnetic
vector potential in the Coulomb gauge). Let $v(x,y)$ denote the Green function of the scalar Laplacian on $M$. The Hamiltonian of the system is defined by

$$H^{(N)} = \sum_j D^2_{A(x_j)} + V_C(x_1, \ldots, x_N; y_1, \ldots, y_K),$$

where

$$V_C(x_1, \ldots, x_N; y_1, \ldots, y_K) = \sum_{1 \leq i < j \leq N} v(x_i, y_j) - \sum_{l=1}^{N} Z_l v(x_i, v_l) + \sum_{1 \leq i < k \leq K} Z_i Z_k v(y_i, y_k).$$

Let

$$\mathcal{E}_{\text{field}}(A) = \Gamma \int_M \|F_A\|^2 \, d\text{vol}_g,$$

where $F_A$ is the curvature of $A$ and $\Gamma > 0$ is a constant. Physically, $\mathcal{E}_{\text{field}}(A)$ is the energy of the magnetic field described by $A$ (for dim $M = 3$).

The problem of stability of non-relativistic matter is the problem of showing that the “energy functional”

$$\mathcal{E}(\psi, A) = \langle \psi, H^{(N)} \psi \rangle_{\mathcal{H}(N)} + \mathcal{E}_{\text{field}}(A),$$

where $\psi \in \mathcal{H}(N)$ has norm 1, is bounded from below by

$$\mathcal{E}(\psi, A) \geq -C(N + K)$$

for a finite constant $C$ only depending on $\Gamma$ and $Z^*$, but independent of $N$ and $K$, provided $\Gamma$ is large enough.

If dim $M \equiv n \leq 2$ stability of matter is not a particularly challenging problem; see [31]. If $n \geq 4$ it cannot be valid, as simple scaling arguments show: $[\mathcal{E}_{\text{field}}] = \text{length}^{n-4}$, $[V_C] = \text{length}^{2-n}$. If $n = 3$, which is the dimension of physical space, then $[\mathcal{E}_{\text{field}}] = [V_C]$, and the problem of proving (4.94) has a physically interesting and mathematically non-trivial solution; see [32] (and refs. given there). In this case there is a critical value, $\Gamma_c$, of $\Gamma$ such that (4.94) holds for $\Gamma > \Gamma_c$ and $Z_s$ small enough (depending on $\Gamma$), but fails for $\Gamma < \Gamma_c$.

The reason why $\Gamma$ must be large enough for (4.94) to hold in $n = 3$ dimensions is that $[\mathcal{E}_{\text{field}}] = [V_C]$ and, for a large class of connections $A$, the Pauli-Dirac operator $D_A$ has zero-modes. For a class of connections $A$ with $\mathcal{E}_{\text{field}}(A) < \infty$, such zero-modes were constructed by Loss and Yau, [33]. For this purpose, these authors studied the equations

$$D_A \psi = 0, \quad c(F_A) = \kappa (\psi \psi^*)_0,$$

where $\psi \in \mathcal{H}_c$, $c$ denotes Clifford multiplication, see eqs. (4.10,11,12), $(\psi \psi^*)_0$ is the traceless part of the matrix $\psi \psi^*$, and $\kappa$ is constant. Using a clever ansatz for $\psi$ and $A$, Loss and Yau exhibited solutions of these equations.

Of course, eqs. (4.95) are related to the famous Seiberg-Witten equations [34], which Witten discovered from the study of supersymmetric Yang-Mills theory on four-dimensional, compact, orientable, smooth Riemannian manifolds (which are automatically spin$^c$). They have invigorated four-dimensional differential topology. Apparently, they
also emerge from problems of stability in the quantum mechanics of non-relativistic electrons (at least when \( \dim M = 3 \))!

As we have seen in Sect. 4.1, the study of the quantum theory of non-relativistic electrons and positrons leads to the discovery of gauge symmetries of the second kind and of supersymmetry. Evidently, the gauge group underlying Pauli’s quantum mechanics of the electron on an \( n \)-dimensional manifold is \( \text{Spin}^c(n) \), which locally is isomorphic to \( \text{Spin}(n) \times U(1) \); \( \text{Spin}(n) \) is the group of rotations in “spin space”, while \( U(1) \) is the group of electromagnetic phase transformations. Local \( U(1) \)-gauge invariance has been used and studied ever since Fock and Weyl discovered it. Invariance of Pauli’s quantum mechanics under local \( \text{Spin}(n) \) rotations has escaped the attention of physicists until recently [35]. Yet, it has important qualitative implications (description of spin-orbit interactions as torsion in the spin connection, quantum mechanical Larmor theorem, Barnett-Einstein-de Haas effect, etc.).

Supersymmetric quantum mechanics and its uses in algebraic topology have been studied extensively [16,22,36]. But the fact that the form of supersymmetric quantum mechanics which the mathematicians have explored so successfully is really Pauli’s quantum theory of the non-relativistic electron with spin and of positronium and the use of supersymmetry arguments in non-relativistic quantum physics have apparently been somewhat under-emphasized.

The example of electrodynamics with non-relativistic, quantum mechanical matter considered above is much more instructive than the reader may have realized. It was recognized by Jordan and Dirac in the late twenties that, in order to construct a theory correctly describing radiation of atoms and molecules and the quantum mechanics of the radiation field (which stood at the origin of quantum theory), one must quantize the electromagnetic field and interpret the \( U(1) \)-connection \( A \) (the electromagnetic vector potential) as an operator-valued distribution on a Hilbert space \( \mathcal{F} \) (the Fock space) of pure state vectors describing photon configurations. The resulting quantum theory consists of the following data:

(i) Its Hilbert space of pure state vectors is given by

\[
\mathcal{H}_{\text{tot}} = \mathcal{H}^{(N)} \otimes \mathcal{F},
\]

where, recall, \( N \) is the number of electrons in the system, which in this theory is conserved.

(ii) The time evolution is generated by (a renormalized version of) the Hamiltonian

\[
H_{\text{tot}} = H^{(N)} + 1 \otimes H_{\text{field}},
\]

where \( H_{\text{field}} \) is the usual free-photon Hamiltonian on \( \mathcal{F} \).

(iii) The algebra of observables \( A_{\text{tot}} \) of the system contains the algebra

\[
A^{(N)} \otimes \mathcal{B},
\]

where \( A^{(N)} \) is the algebra of smooth functions on \( M \times N \), and \( \mathcal{B} \) is an algebra of bounded functions of the quantized magnetic field (smeared out with test functions).
For the sake of backing up this tale with mathematically precise results — see e.g. [37] — one must assume that \( \dim M \leq 3 \) (although, even under this hypothesis, there remain open problems).

Our purpose in mentioning this example is not to now engage in a discussion of the beautiful and very rich physics described by quantum electrodynamics with non-relativistic matter, as defined in (i) through (iii) above. We just want to make the following crucial point: As we have learned in Sect. 4.1, the electromagnetic vector potential \( A \) (the “virtual” U(1)-connection) is part of the geometric data (associated with spin \( c \) manifolds) on which the quantum mechanics of non-relativistic electrons hinges. Apparently, the physics of natural phenomena forces us to view the connection \( A \) as operator-valued, and to subject its curvature \( F_A \) describing the electric field \( E \) (time-derivative of \( A \)) and the magnetic field \( B \) (spatial derivatives of \( A \)) to the uncertainty relations (2.3) of Section 2. It is the quantization of the electromagnetic field that makes excited atoms emit light spontaneously. The model of quantum electrodynamics with non-relativistic matter described in (i–iii) above enables us to describe such (and other) phenomena in a physically acceptable and mathematically rigorous way (at least if the Hamiltonian of the theory is regularized at short distances). The example is interesting in a second respect: Using an operator-theoretic version of renormalization group methods, see [37], one can make precise the claim that classical behaviour, in particular classical geometry, reappears at very large distance (and long time) scales.

Let \( E \) be a Hermitian vector bundle over \( M \) associated to a principal \( G \)-bundle, where \( G \) is a compact Lie group (e.g. SU(2), SU(3)). Let \( V \) denote the fiber of \( E \). We can equip \( E \) with a connection \( \nabla^E \) inherited from a connection on the principal \( G \)-bundle over \( M \). A heavy quark in a hadron can be viewed as a variant of a non-relativistic electron described by data consisting of \( \mathcal{H}_q \), the space of square-integrable sections of a Hermitian vector bundle over \( M \) with fiber isomorphic to \( W \otimes V \), where \( W \cong \mathbb{C}^\left\lfloor n \right\rfloor \). This bundle is equipped with a connection determined by \( \nabla^S \) and \( \nabla^E \). This connection determines a Pauli-Dirac operator \( D \), the square of which gives rise to a Hamiltonian \( H \) as in eq. (4.21). In order to arrive at a correct quantum mechanical description of heavy quarks bound in a hadron, one must generalize the theory to encompass \( N = 1, 2, 3, \ldots \) quarks, and one must quantize \( \nabla^E \): The components of \( \nabla^E \) are interpreted as operator-valued distributions. The resulting “quantum chromodynamics” (with non-relativistic quarks) is not so well understood, yet.

Our discussion makes it tempting to imagine that, in nature, all the data characterizing the \( K \)-theory and geometry of physical space-(time) must be quantized, including the spin connection and the Riemannian metric. This is the topic of quantum gravity, a quantum theory that, to date, is not well understood and hence is really not a theory in the mathematical sense, yet. But we are able to guess some of its crude features, and these compel one to generalize classical to non-commutative geometry, as described in Section 5.

Our discussion in Sects. 4.1 and 4.2 has made it clear that, in the context of finite-dimensional classical manifolds, (globally) supersymmetric quantum mechanics — as it emerges from the study of non-relativistic electrons and positrons — is just another name for classical differential topology and geometry. Actually, this is a general fact: Global supersymmetry, whether in quantum mechanics or in quantum field theory, is just another name for the differential topology and geometry of (certain) spaces.
In the following, we indicate why globally supersymmetric quantum field theory, too, is nothing more than geometry of infinite-dimensional spaces. However, once we pass from supersymmetric quantum mechanics to quantum field theory, there are surprises.

A non-linear $\sigma$-model is a field theory of maps from a parameter “space-time” $\Sigma$ to a target space $M$. Under suitable conditions on $M$, a non-linear $\sigma$-model can be extended to a supersymmetric theory, (see e.g. [38]). One tends to imagine that such models can be quantized. When $\Sigma$ is the real line $\mathbb{R}$, this is indeed possible, and one recovers supersymmetric quantum mechanics in the sense explained in the previous sections. When $\Sigma = S^1 \times \mathbb{R}$, there is hope that quantization is possible, and one obtains an analytic tool to explore the infinite-dimensional geometry of loop space $M^{S^1}$. When $\Sigma = L \times \mathbb{R}$ with $\dim L \geq 2$, the situation is far less clear, but a supersymmetric non-linear $\sigma$-model with parameter space-time $L \times \mathbb{R}$ could be used to explore the geometry of $M^L$ — i.e., of poorly understood infinite-dimensional manifolds.

A (quantized) supersymmetric $\sigma$-model with $n$ global supersymmetries and with parameter space $\Sigma = T^d \times \mathbb{R}$, where $T^d$ is a $d$-dimensional torus, formally determines a model of supersymmetric quantum mechanics by dimensional reduction: The supersymmetry algebra of the non-linear $\sigma$-model with this $\Sigma$ contains the algebra of infinitesimal translations on $\Sigma$; by restricting the theory to the Hilbert sub-space that carries the trivial representation of the group of translations of $T^d$ one obtains a model of supersymmetric quantum mechanics. The supersymmetry algebra can be reduced to this “zero-momentum” subspace, and the restricted algebra is of the form discussed in Sect. 4.1. Starting from $\hat{n}$ supersymmetries and a $d + 1$-dimensional parameter space-time $\Sigma$, one ends up with a model of $N = (n, n)$ supersymmetric quantum mechanics where $n = \hat{n}$ for $d = 1, 2, n = 2\hat{n}$ for $d = 3, 4, n = 4\hat{n}$ for $d = 5, 6$, and $n = 8\hat{n}$ for $d = 7, 8$ (see e.g. [39]). The resulting supersymmetric quantum mechanics (when restricted to an even smaller subspace of “zero modes”) is expected to encode the geometry of the target space $M$ — in, roughly speaking, the sense outlined in Sect. 4.1. From what we have learned there, it follows at once that target spaces of $\sigma$-models with many supersymmetries or with a high-dimensional parameter space-time must have very special geometries.

This insight is not new. It has been gained in a number of papers, starting in the early eighties with work of Alvarez-Gaumé and Freedman, see [38]. Supersymmetric quantum mechanics is a rather old idea, too, beginning with papers by Witten [15,16] — which, as is well known, had a lot of impact on mathematics. In later works, “supersymmetry proofs” of the index theorem were given [22]. The reader may find comments on the history of global supersymmetry e.g. in [40].

A (quantum) field theory of Bose fields can always be thought of as a $\sigma$-model (linear or non-linear), i.e., as a theory of maps from a parameter “space-time” $\Sigma = L \times \mathbb{R}$ to a target space $M$. At the level of classical field theory, we may attempt to render such a model globally supersymmetric, and then to quantize it. As we have discussed above, the resulting quantum field theory — if it exists — provides us with the spectral data to explore the geometry of what one might conjecture to be some version of the formal infinite-dimensional manifold $M^L$. It may happen that the quantum field theory exhibits some form of invariance under re-parameterizations of parameter space-time $\Sigma$ (though such an invariance can be destroyed by anomalies, even if present at the classical level). However, when $\Sigma = \mathbb{R}$, re-parameterization invariance can be imposed; when e.g. $\Sigma = S^1 \times \mathbb{R}$, it leads us to the tree-level formulation of first-quantized string theory. Let us be content with “conformal invariance” and study (supersymmetric) conformal
σ-models of maps from parameter space-time $\Sigma = S^1 \times \mathbb{R}$ to a target space $M$ which, for concreteness, we choose to be a smooth, compact Riemannian manifold without boundary, at the classical level. An example would be $M = G$, some compact (simply laced) Lie group. The corresponding field theory is the supersymmetric Wess-Zumino-Witten model [41], which is surprisingly well understood. Thanks to the theory of Kac-Moody algebras and centrally extended loop groups, see e.g. [42], its quantum theory is under fairly complete mathematical control and provides us with the spectral data of some $N = (1, 1)$ supersymmetric quantum theory (related to the example discussed in Sect. 4.2 with spontaneously broken supersymmetry; see e.g. [24]). We might expect that these spectral data encode the geometry of the loop space $G^{S^1}$ over $G$. The surprise is, though, that they encode the geometry of loop space over a quantum deformation of $G$ (where the deformation parameter depends on the level $k$ of the Wess-Zumino-Witten model in such a way that, formally, $k \to \infty$ corresponds to the classical limit). We expect that this is an example of a general phenomenon: Quantized supersymmetric σ-models with parameter space-time $\Sigma$ of dimension $d \geq 2$ — assuming that they exist — tend to provide us with the spectral data of a supersymmetric quantum theory which encodes the geometry of a “quantum deformation” of the target space of the underlying classical σ-model.

This is one reason why quantum physics forces us to go beyond classical differential geometry. A second, more fundamental topic which calls for “quantum geometry” is the outstanding problem of unifying the quantum theory of matter with the theory of gravity within a theory of quantum gravity, as discussed in Section 3 above.

5 Supersymmetry and non-commutative geometry

In this section, we attempt to describe the geometry of generalized spaces, such as discrete sets, graphs, quantum phase spaces, and more general non-commutative spaces, in such a way that the spin$^c$, Riemannian, complex, etc. geometries of classical manifolds emerge as special cases. This is the subject that Connes calls non-commutative geometry (NCG), see [5,43].

Our approach to NCG is inspired by [15,16,5,43]; we follow the presentation in [18], where the reader finds further details (in particular on classical geometry). The emphasis is put on the general structure of the theory and on key ideas, rather than on technical details.

NCG is not a particularly well developed theory, yet, and the number of well understood examples is quite limited. Typically, they involve discrete sets, the quantum phase spaces over discrete sets, and deformation quantizations of Kähler manifolds.

There are three starting points for generalizing classical geometry to NCG:

(1) Geometry of non-commutative metric spaces. Here we start from spectral data $(\mathcal{A}, \mathcal{H}, \Delta)$ where

(i) $\mathcal{H}$ is a separable Hilbert space;
(ii) $\mathcal{A}$ is a $C^*$–algebra faithfully represented on $\mathcal{H}$;
(iii) $\Delta$ is a self-adjoint operator on $\mathcal{H}$ such that $\exp(-\varepsilon \Delta)$ is trace class, for arbitrary $\varepsilon > 0$; there exists a norm-dense subalgebra $\mathcal{A}$ of $\mathcal{A}$ such that the
operator

\[
\frac{1}{2} \left( \triangle a^2 + a^2 \triangle \right) - a \triangle a
\]

is bounded for an arbitrary \( a \in \mathcal{A}^0 \).

This structure has been described and studied in [24]; see also Section 2. We shall not pursue it here, although this approach leads to interesting mathematical problems.

(2) \textit{Spin}^c geometry, cast in the form of \( N = 1 \) supersymmetric quantum theory, which is inspired by Pauli’s quantum mechanics of the non-relativistic electron with spin. This is Connes’ starting point [5].

(3) \textit{Riemannian geometry}, cast in the form of \( N = (1, 1) \) or \( N = (1, \bar{1}) \) supersymmetric quantum theory, which is inspired by Pauli’s quantum mechanics of non-relativistic positronium. This approach is described in some detail in [18].

In classical \textit{spin}^c geometry, we can always pass from (2) to (3) by considering the tensor product bundle of the spinor- and the charge-conjugate spinor bundle; in NCG it may not always be possible to pass from (2) to (3), and this justifies that we describe both approaches.

An example of NCG (the non-commutative torus) will be described in Section 6 and applications to string- and membrane theory in Section 7.

5.1 \textit{Spin}^c non-commutative geometry

Our starting point is a natural generalization of the \( N = 1 \) supersymmetric quantum theory of a non-relativistic electron described in Sect. 4.1.

1) The spectral data of \textit{spin}^c NCG.

\textbf{Definition A.} A \textit{spin}^c non-commutative space is described by \( N = 1 \) spectral data \((\mathcal{A}, \mathcal{H}, D, \gamma)\) with the following properties:

(1) \( \mathcal{H} \) is a separable Hilbert space;

(2) \( \mathcal{A} \) is a unital \( * \)-algebra faithfully represented on \( \mathcal{H} \);

(3) \( D \) is a self-adjoint operator on \( \mathcal{H} \) such that

\begin{enumerate}
    \item i) for each \( a \in \mathcal{A} \), the commutator \( [D,a] \) defines a \textit{bounded} operator on \( \mathcal{H} \),
    \item ii) the operator \( \exp(-\varepsilon D^2) \) is trace class for all \( \varepsilon > 0 \);
\end{enumerate}

(4) \( \gamma \) is a \( \mathbb{Z}_2 \)-grading on \( \mathcal{H} \), i.e., \( \gamma = \gamma^* = \gamma^{-1} \), such that

\[
\{ \gamma, D \} = 0, \quad [\gamma, a] = 0, \quad \text{for all} \quad a \in \mathcal{A}.
\]
In NCG, $\mathcal{A}$ plays the role of the “algebra of functions over a non-commutative space”. The existence of a unit in $\mathcal{A}$ and property 3) ii) mean that we are only considering “compact” non-commutative spaces.

Note that if the Hilbert space $\mathcal{H}$ is infinite-dimensional, condition 3) ii) implies that the operator $D$ is unbounded. By analogy with classical differential geometry, $D$ is interpreted as a (generalized) Dirac operator.

Also note that the fourth condition in Definition A does not impose any restriction on $N = 1$ spectral data: In fact, given a triple $(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$ satisfying properties (1–3) above, we can define a set of $N = 1$ even spectral data $(\mathcal{A}, \mathcal{H}, D, \gamma)$ by setting

$$
\mathcal{H} = \tilde{\mathcal{H}} \otimes \mathbb{C}^2, \quad \mathcal{A} = \tilde{\mathcal{A}} \otimes 1_2, \\
D = \tilde{D} \otimes \tau_1, \quad \gamma = 1_{\tilde{\mathcal{H}}} \otimes \tau_3,
$$

where $\tau_i$ are the Pauli matrices acting on $\mathbb{C}^2$. See [5] for further background and motivation.

2) Differential forms.

Given a unital $*$-algebra $\mathcal{A}$, as in 1), let $\Omega^\bullet(\mathcal{A})$ denote the universal unital, graded, differential algebra of “forms” constructed by Connes and Karoubi [44], which can be described as follows:

$$
\Omega^\bullet(\mathcal{A}) = \bigoplus_{n=0}^{\infty} \Omega^n(\mathcal{A}),
$$

where $\Omega^n(\mathcal{A})$ is spanned by elements $\alpha$ of the form

$$
\alpha = \sum_j a_j^0 \delta a_j^1 \cdots \delta a_j^n, \quad (5.1)
$$

with $a_j^i \in \mathcal{A}$ for all $i$ and $j$, and the “derivation” $\delta$ has the following properties:

(i) $\delta$ is linear, and $(\delta a)^* := -\delta a^*$, which makes $\Omega^\bullet(\mathcal{A})$ into a $*$-algebra;

(ii) $\delta$ satisfies the Leibniz rule

$$
\delta(ab) = (\delta a)b + a(\delta b),
$$

for all $a, b$ in $\mathcal{A}$; in particular $\delta 1 = 0$, where 1 is the unit element in $\mathcal{A}$;

(iii) $\delta^2 = 0$.

Given spectral data $(\mathcal{A}, \mathcal{H}, D, \gamma)$, as in Definition A, we define a $*$-homomorphism $\pi$ from $\Omega^\bullet(\mathcal{A})$ to $B(\mathcal{H})$ by setting

$$
\pi(a) := a, \quad \pi(\delta a) := [D,a].
$$

A graded $*$-ideal $J$ of $\Omega^\bullet(\mathcal{A})$ is defined by

$$
J = \bigoplus_{n=0}^{\infty} J^n, \quad J^n := \ker \pi \big|_{\Omega^n(\mathcal{A})}, \quad (5.2)
$$
Since in general $J$ is not a differential ideal, the graded quotient $\Omega^\bullet(A)/J$ does not define a differential algebra. However, it is easy to show [5] that the graded sub-complex

$$J + \delta J := \bigoplus_{n=0}^{\infty} \left( J^n + \delta J^{n-1} \right),$$

with $J^{-1} := \{0\}$, is a two-sided graded differential $\ast$–ideal of $\Omega^\bullet(A)$. (This follows from $\delta^2 = 0$ and from the Leibniz rule.)

The unital, graded, differential $\ast$–algebra of differential forms, $\Omega^\bullet_D(A)$, is defined by

$$\Omega^\bullet_D(A) = \bigoplus_{n=0}^{\infty} \Omega^n_D(A),$$

where

$$\Omega^n_D(A) := \Omega^n(A)/(J^n + \delta J^{n-1}).$$

(5.3)

Each subspace $\Omega^n_D(A)$ is a bi-module over $A = \Omega^0_D(A)$.

An $n$–form on $\mathcal{H}$ is an equivalence class of bounded operators on $\mathcal{H}$: For $[\alpha] \in \Omega^n_D(A)$,

$$\pi([\alpha]) = \pi(\alpha) + \pi(\delta J^{n-1}).$$

The image of $\Omega^\bullet_D(A)$ under $\pi$ is $\mathbb{Z}_2$–graded:

$$\pi(\Omega^\bullet_D(A)) = \pi\left( \bigoplus_{n=0}^{\infty} \Omega^{2n}_D(A) \right) \oplus \pi\left( \bigoplus_{n=0}^{\infty} \Omega^{2n+1}_D(A) \right),$$

where elements of the first summand on the r.s. commute with $\gamma$, while elements of the second summand anti-commute with $\gamma$ (where $\gamma$ is the $\mathbb{Z}_2$-grading of Definition A).

3) Integration

Property 3) ii) of the Dirac operator in Definition A allows us to define a notion of integration over a non-commutative space in the same way as in the classical case. Note that, for certain sets of $N = 1$ spectral data, we could use the Dixmier trace, as Connes originally proposed; but the definition given below, first introduced in [45], works in greater generality. Moreover, it is closer to constructions in quantum field theory.

**Definition B.** The integral over the non-commutative space described by the $N = 1$ spectral data $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is a state $f$ on $\pi(\Omega^\bullet(A))$ defined by

$$f : \begin{cases} \pi(\Omega^\bullet(A)) & \longrightarrow \mathbb{C} \\ \omega & \longmapsto f(\omega) := \lim_{\varepsilon \to 0^+} \frac{\text{Tr}_\mathcal{H}(\omega e^{-\varepsilon D^2})}{\text{Tr}_\mathcal{H}(e^{-\varepsilon D^2})}, \end{cases}$$

where $\lim_{\varepsilon \to 0^+}$ denotes some limiting procedure making the functional $f$ linear and positive semi-definite; existence of such a procedure can be shown analogously to [5,46], where the Dixmier trace is discussed.

For the integral $f$ to be a useful tool, we need an additional property that must be checked in each example:
Assumption A. The state $\int$ on $\pi(\Omega^*(A))$ is cyclic, i.e.,

$$\int \omega \eta^* = \int \eta^* \omega$$

for all $\omega, \eta \in \pi(\Omega^*(A))$. (A weaker form is to only assume that $\int \omega a = \int a \omega$, for all $a \in A$, $\omega \in \pi(\Omega^*(A))$.)

The state $\int$ determines a positive semi-definite sesqui-linear form on $\Omega^*(A)$ by setting

$$(\omega, \eta) := \int \pi(\omega) \pi(\eta)^*$$

for all $\omega, \eta \in \Omega^*(A)$. In the formulas below, we will often drop the representation symbol $\pi$ under the integral, as there is no danger of confusion.

Note that the commutation relations of the grading $\gamma$ with the Dirac operator imply that forms of odd degree are orthogonal to those of even degree with respect to $(\cdot, \cdot)$.

By $K^k$ we denote the kernel of this sesqui-linear form restricted to $\Omega^k(A)$. More precisely, we set

$$K := \bigoplus_{k=0}^{\infty} K^k, \quad K^k := \{ \omega \in \Omega^k(A) \mid (\omega, \omega) = 0 \}.$$  \hspace{1cm} (5.5)

Obviously, $K^k$ contains the ideal $J^k$ defined in eq. (5.2); in the classical case they coincide. Assumption A is needed to show that $K$ is a two-sided graded $^*$–ideal of the algebra of universal forms, too, so that we can pass to the quotient algebra, see [18]. We now define

$$\tilde{\Omega}^*(A) := \bigoplus_{k=0}^{\infty} \tilde{\Omega}^k(A), \quad \tilde{\Omega}^k(A) := \Omega^k(A)/K^k.$$  \hspace{1cm} (5.6)

The sesqui-linear form $(\cdot, \cdot)$ defines a positive definite scalar product on $\tilde{\Omega}^k(A)$, and we denote by $\tilde{\mathcal{H}}^k$ the Hilbert space completion of this space with respect to the scalar product,

$$\tilde{\mathcal{H}}^* := \bigoplus_{k=0}^{\infty} \tilde{\mathcal{H}}^k, \quad \tilde{\mathcal{H}}^k := \tilde{\Omega}^k(A)^{(\cdot, \cdot)}.$$  \hspace{1cm} (5.7)

$\tilde{\mathcal{H}}^k$ is to be interpreted as the space of square-integrable $k$–forms. Note that $\tilde{\mathcal{H}}^*$ does not quite coincide with the Hilbert space that would arise from a GNS construction using the state $\int$ on $\tilde{\Omega}^*(A)$: Whereas in $\mathcal{H}^*$, orthogonality of forms of different degree is installed by definition, there may occur mixings among forms of even degrees (or among odd forms) in the GNS Hilbert space.

One now shows that the space $\tilde{\Omega}^*(A)$ is a unital graded $^*$–algebra. For any $\omega \in \tilde{\Omega}^k(A)$, the left and right actions of $\omega$ on $\tilde{\Omega}^p(A)$, with values in $\tilde{\Omega}^{p+k}(A)$,

$$m_L(\omega) \eta := \omega \eta, \quad m_R(\omega) \eta := \eta \omega,$$

are continuous in the norm given by $(\cdot, \cdot)$.  \hspace{1cm} (5.8)
Since the algebra $\hat{\Omega}^\bullet(A)$ may fail to be differential, we introduce the unital graded differential $^\ast$–algebra of square-integrable differential forms $\hat{\Omega}_D^\bullet(A)$ as the graded quotient of $\Omega^\bullet(A)$ by $K + \delta K$,

$$\hat{\Omega}_D^\bullet(A) := \bigoplus_{k=0}^{\infty} \hat{\Omega}_D^k(A), \quad \hat{\Omega}_D^k(A) := \Omega^k(A) / (K^k + \delta K^{k-1}) \cong \hat{\Omega}^k(A)/\delta K^{k-1}. \quad (5.8)$$

Note that we can regard the $A$–bi-module $\hat{\Omega}_D^\bullet(A)$ as a “smaller version” of $\Omega_D^\bullet(A)$, in the sense that there exists a projection from the latter onto the former.

In the classical case, differential forms can be identified with the orthogonal complement of $Cl^{(k-2)}$ within $Cl^{(k)}$, where $Cl^{(k)}$ denotes the $k$ th subspace in the filtration of the space of sections of the Clifford bundle, see [5,18]. Now, we use the scalar product $(\cdot, \cdot)$ on $\tilde{\mathcal{H}}^k$ to introduce, for each $k \geq 1$, the orthogonal projection

$$P_{\delta K^{k-1}} : \tilde{\mathcal{H}}^k \longrightarrow \tilde{\mathcal{H}}^k \quad (5.9)$$

onto the image of $\delta K^{k-1}$ in $\tilde{\mathcal{H}}^k$, and we set

$$\omega^\perp := (1 - P_{\delta K^{k-1}}) \omega \quad \in \tilde{\mathcal{H}}^k \quad (5.10)$$

for each element $[\omega] \in \hat{\Omega}_D^k(A)$. This allows us to define a positive definite scalar product on $\hat{\Omega}_D^k(A)$ via the representative $\omega^\perp$:

$$([\omega], [\eta]) := (\omega^\perp, \eta^\perp) \quad (5.11)$$

for all $[\omega], [\eta] \in \hat{\Omega}_D^k(A)$. In the classical case, this is just the usual inner product on the space of square-integrable $k$–forms.

4) Vector bundles and Hermitian structures

We follow the algebraic formulation of classical differential geometry, in order to generalize the notion of a vector bundle to the non-commutative case.

**Definition C.** [5] A vector bundle $\mathcal{E}$ over the non-commutative space described by the $N = 1$ spectral data $(A, \mathcal{H}, D, \gamma)$ is a finitely generated, projective left $A$–module.

Recall that a module $\mathcal{E}$ is projective if there exists another module $\mathcal{F}$ such that the direct sum $\mathcal{E} \oplus \mathcal{F}$ is free, i.e., $\mathcal{E} \oplus \mathcal{F} \cong A^n$ as left $A$–modules, for some $n \in \mathbb{N}$. Since $A$ is an algebra, every $A$–module is a vector space; therefore, left $A$–modules are representations of the algebra $A$, and $\mathcal{E}$ is projective iff there exists a module $\mathcal{F}$ such that $\mathcal{E} \oplus \mathcal{F}$ is isomorphic to a multiple of the left-regular representation.

By Swan’s Lemma [47], a finitely generated, projective left module corresponds, in the commutative case, to the space of sections of a vector bundle.

It is straightforward to define the notion of a Hermitian structure over a vector bundle:

**Definition D.** [5] A Hermitian structure over a vector bundle $\mathcal{E}$ is a sesqui-linear map (linear in the first argument)

$$\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \longrightarrow A$$

such that for all $a, b \in A$ and all $s, t \in \mathcal{E}$
1) \( \langle as, bt \rangle = a \langle s,t \rangle b^*; \)

2) \( \langle s,s \rangle \geq 0; \)

3) the \( \mathcal{A} \)-linear map

\[
g : \begin{cases} 
E &\rightarrow \mathcal{E}_R^* \\
 s &\mapsto \langle s, \cdot \rangle 
\end{cases},
\]

where \( \mathcal{E}_R^* := \{ \phi \in \text{Hom}(\mathcal{E}, \mathcal{A}) | \phi(as) = \phi(s)a^* \} \), is an isomorphism of left \( \mathcal{A} \)-modules, i.e., \( g \) can be regarded as a metric on \( \mathcal{E} \).

In the second condition, the notion of positivity in \( \mathcal{A} \) is simply inherited from the algebra \( \mathcal{B}(\mathcal{H}) \) of all bounded operators on the Hilbert space \( \mathcal{H} \).

5) Generalized Hermitian structure on \( \tilde{\Omega}^k(\mathcal{A}) \)

It turns out that the \( \mathcal{A} \)-bi-modules \( \tilde{\Omega}^k(\mathcal{A}) \) carry Hermitian structures in a slightly generalized sense. Let \( \mathcal{A}'' \) be the weak closure of the algebra \( \mathcal{A} \) acting on \( \mathcal{H}^0 \), i.e., \( \mathcal{A}'' \) is the von Neumann algebra generated by \( \tilde{\Omega}^0(\mathcal{A}) \) acting on the Hilbert space \( \mathcal{H}^0 \).

**Theorem** [45,18]. There is a canonically defined sesqui-linear map

\[
\langle \cdot, \cdot \rangle_D : \tilde{\Omega}^k(\mathcal{A}) \times \tilde{\Omega}^k(\mathcal{A}) \rightarrow \mathcal{A}''
\]

such that, for all \( a, b \in \mathcal{A} \) and all \( \omega, \eta \in \tilde{\Omega}^k(\mathcal{A}) \),

1) \( \langle a\omega, b\eta \rangle_D = a \langle \omega, \eta \rangle_D b^*; \)

2) \( \langle \omega, \omega \rangle_D \geq 0; \)

3) \( \langle \omega a, \eta \rangle_D = \langle \omega, \eta a^* \rangle_D. \)

We call \( \langle \cdot, \cdot \rangle_D \) a *generalized Hermitian structure* on \( \tilde{\Omega}^k(\mathcal{A}) \). It is the non-commutative analogue of the Riemannian metric on the bundle of differential forms. Note that \( \langle \cdot, \cdot \rangle_D \) takes values in \( \mathcal{A}'' \), and thus property 3) of Definition D is not directly applicable. For the proof of the theorem see [45,18].

6) Connections

**Definition E.** A *connection* \( \nabla \) on a vector bundle \( \mathcal{E} \) over a non-commutative space is a \( \mathbb{C} \)-linear map

\[
\nabla : \mathcal{E} \rightarrow \tilde{\Omega}^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}
\]

such that

\[
\nabla(as) = \delta a \otimes s + a \nabla s
\]

for all \( a \in \mathcal{A} \) and all \( s \in \mathcal{E} \).

Given a vector bundle \( \mathcal{E} \), we define a space of \( \mathcal{E} \)-valued differential forms by

\[
\tilde{\Omega}^\bullet(\mathcal{E}) := \tilde{\Omega}^\bullet(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E};
\]

if \( \nabla \) is a connection on \( \mathcal{E} \) then it extends uniquely to a \( \mathbb{C} \)-linear map, again denoted \( \nabla \),

\[
\nabla : \tilde{\Omega}^\bullet(\mathcal{E}) \rightarrow \tilde{\Omega}^{\bullet+1}(\mathcal{E})
\] (5.12)
such that
\[ \nabla(\omega s) = (\delta \omega) s + (-1)^k \omega \nabla s \] (5.13)
for all \( \omega \in \tilde{\Omega}^k_D(\mathcal{A}) \) and all \( s \in \tilde{\Omega}^*_D(\mathcal{E}) \).

**Definition F.** The *curvature* of a connection \( \nabla \) on a vector bundle \( \mathcal{E} \) is given by
\[
R(\nabla) = - \nabla^2 : \mathcal{E} \rightarrow \tilde{\Omega}^2_D(\mathcal{A}) \otimes \mathcal{A} \mathcal{E}.
\]
Note that the curvature extends to a map
\[
R(\nabla) : \tilde{\Omega}^*_D(\mathcal{E}) \rightarrow \tilde{\Omega}^{*+2}_D(\mathcal{E})
\]
which is left \( \mathcal{A} \)-linear, as follows from eq. (5.13) and Definition E.

**Definition G.** A connection \( \nabla \) on a Hermitian vector bundle \((\mathcal{E}, \langle \cdot, \cdot \rangle)\) is called *unitary* if
\[
\delta \langle s, t \rangle = \langle \nabla s, t \rangle - \langle s, \nabla t \rangle
\]
for all \( s, t \in \mathcal{E} \), where the r.s. of this equation is defined by
\[
\langle \omega \otimes s, t \rangle = \omega \langle s, t \rangle, \quad \langle s, \eta \otimes t \rangle = \langle s, t \rangle \eta^*\] (5.14)
for all \( \omega, \eta \in \tilde{\Omega}^1_D(\mathcal{A}) \) and all \( s, t \in \mathcal{E} \).

7) **Riemannian curvature and torsion**
Throughout this subsection, we make three additional assumptions which limit the generality of our results, but turn out to be fulfilled in interesting examples.

**Assumption B.** We assume that the \( N = 1 \) spectral data under consideration have the following additional properties:

1) \( K^0 = 0 \). (This implies that \( \tilde{\Omega}^0_D(\mathcal{A}) = \mathcal{A} \) and \( \tilde{\Omega}^1_D(\mathcal{A}) = \tilde{\Omega}^1(\mathcal{A}) \); thus \( \tilde{\Omega}^1_D(\mathcal{A}) \) carries a generalized Hermitian structure.)

2) \( \tilde{\Omega}^1_D(\mathcal{A}) \) is a vector bundle, called the *cotangent bundle over* \( \mathcal{A} \). (\( \tilde{\Omega}^1_D(\mathcal{A}) \) is always a left \( \mathcal{A} \)-module. Here, we assume, in addition, that it is \emph{finitely generated} and \emph{projective}.)

3) The generalized metric \( \langle \cdot, \cdot \rangle_D \) on \( \tilde{\Omega}^1_D(\mathcal{A}) \) defines an isomorphism of left \( \mathcal{A} \)-modules between \( \tilde{\Omega}^1_D(\mathcal{A}) \) and the space of \( \mathcal{A} \)-anti-linear maps from \( \tilde{\Omega}^1_D(\mathcal{A}) \) to \( \mathcal{A} \), i.e., for each \( \mathcal{A} \)-anti-linear map
\[
\phi : \tilde{\Omega}^1_D(\mathcal{A}) \rightarrow \mathcal{A}
\]
with \( \phi(a\omega) = \phi(\omega)a^* \), for all \( \omega \in \tilde{\Omega}^1_D(\mathcal{A}) \) and all \( a \in \mathcal{A} \), there is a unique \( \eta_{\phi} \in \tilde{\Omega}^1_D(\mathcal{A}) \) with
\[
\phi(\omega) = \langle \eta_{\phi}, \omega \rangle_D.
\]
If \( N = 1 \) spectral data \((\mathcal{A}, \mathcal{H}, D, \gamma)\) satisfy these assumptions, we are able to define non-commutative generalizations of classical notions like curvature and torsion. Whereas torsion and Riemann curvature can be introduced whenever \( \tilde{\Omega}^1_D(\mathcal{A}) \) is a vector bundle, the third assumption above will provide a substitute for the procedure of “contracting indices” leading to Ricci and scalar curvature.

**Definition H.** Let \( \nabla \) be a connection on the cotangent bundle \( \tilde{\Omega}^1_D(\mathcal{A}) \) over a non-commutative space \((\mathcal{A}, \mathcal{H}, D, \gamma)\) satisfying Assumption B. The torsion of \( \nabla \) is the \( \mathcal{A} \)-linear map

\[
T(\nabla) := \delta - m \circ \nabla : \tilde{\Omega}^1_D(\mathcal{A}) \rightarrow \tilde{\Omega}^2_D(\mathcal{A})
\]

where \( m : \tilde{\Omega}^1_D(\mathcal{A}) \otimes_A \tilde{\Omega}^1_D(\mathcal{A}) \rightarrow \tilde{\Omega}^2_D(\mathcal{A}) \) denotes the product of 1-forms in \( \tilde{\Omega}^1_D(\mathcal{A}) \).

Using the definition of a connection, \( \mathcal{A} \)-linearity of torsion is easy to verify. In analogy to the classical case, a unitary connection \( \nabla \) with \( T(\nabla) = 0 \) is called a Levi-Civita connection. Note, however, that for a given set of non-commutative spectral data, there may be several Levi-Civita connections — or none at all.

Since we assume that \( \tilde{\Omega}^1_D(\mathcal{A}) \) is a vector bundle, we can define the Riemannian curvature of a connection \( \nabla \) on the cotangent bundle as a specialization of Definition F. To proceed further, we make use of part 2) of Assumption B, which implies that there exists a finite set of generators \( \{ E^A \} \) of \( \tilde{\Omega}^1_D(\mathcal{A}) \) and an associated “dual basis” \( \{ \varepsilon_A \} \subset \tilde{\Omega}^1_D(\mathcal{A})^* \),

\[
\tilde{\Omega}^1_D(\mathcal{A})^* := \left\{ \phi : \tilde{\Omega}^1_D(\mathcal{A}) \rightarrow \mathcal{A} \mid \phi(a\omega) = a\phi(\omega) \right\}
\]

such that each \( \omega \in \tilde{\Omega}^1_D(\mathcal{A}) \) can be written as \( \omega = \varepsilon_A(\omega)E^A \), see e.g. [48]. Since the curvature is \( \mathcal{A} \)-linear, there is a family of elements \( \{ R^A_B \} \subset \tilde{\Omega}^2_D(\mathcal{A}) \) with

\[
R(\nabla) = \varepsilon_A \otimes R^A_B \otimes E^B ;
\]

(5.15)

here and in the following the summation convention is used. Put differently, we have applied the canonical isomorphism of vector spaces

\[
\text{Hom}_A\left( \tilde{\Omega}^1_D(\mathcal{A}), \tilde{\Omega}^2_D(\mathcal{A}) \otimes_A \tilde{\Omega}^1_D(\mathcal{A}) \right) \cong \tilde{\Omega}^1_D(\mathcal{A})^* \otimes_A \tilde{\Omega}^2_D(\mathcal{A}) \otimes_A \tilde{\Omega}^1_D(\mathcal{A})
\]

which exists because \( \tilde{\Omega}^1_D(\mathcal{A}) \) is projective — and chosen explicit generators \( E^A, \varepsilon_A \).

Then we have that \( R(\nabla)\omega = \varepsilon_A(\omega)R^A_B \otimes E^B \) for any 1-form \( \omega \in \tilde{\Omega}^1_D(\mathcal{A}) \).

Note that although the components \( R^A_B \) need not be unique, the tensor on the r.s. of eq. (5.15) is well-defined. Likewise, the Ricci and the scalar curvature, to be introduced below, will be invariant combinations of those components, as long as we make sure that all maps we use have the correct “tensorial properties” with respect to the \( \mathcal{A} \)-action.

The last part of Assumption B guarantees that to each \( \varepsilon_A \) there exists a unique 1-form \( e_A \in \tilde{\Omega}^1_D(\mathcal{A}) \) such that

\[
\varepsilon_A(\omega) = \langle \omega, e_A \rangle_D
\]

for all \( \omega \in \tilde{\Omega}^1_D(\mathcal{A}) \). Every such \( e_A \) determines a bounded operator \( m_L(e_A) : \mathcal{H}^1 \rightarrow \mathcal{H}^2 \) acting on \( \mathcal{H}^1 \) by left multiplication with \( e_A \). The adjoint of this operator w.r.t. the scalar product \( \langle \cdot, \cdot \rangle \) on \( \mathcal{H}^* \) is denoted by

\[
e^a_A : \mathcal{H}^2 \rightarrow \mathcal{H}^1 .
\]

(5.16)
\( e^\text{ad}_A \) is a map of right \( \mathcal{A} \)-modules, and it is easy to see that the correspondence \( \varepsilon_A \mapsto e^\text{ad}_A \) is right \( \mathcal{A} \)-linear: For all \( b \in \mathcal{A} \), \( \omega \in \tilde{\Omega}^1_\Omega(D) \), we have that
\[
(\varepsilon_A \cdot b)(\omega) = \varepsilon_A(\omega) \cdot b = \langle \omega, e_A \rangle b = \langle \omega, b^* e_A \rangle ,
\]
and, furthermore, for all \( \xi_1 \in \tilde{\mathcal{H}}^1 \), \( \xi_2 \in \tilde{\mathcal{H}}^2 \),
\[
(b^* e_A(\xi_1), \xi_2) = (e_A(\xi_1), b \xi_2) = (\xi_1, e_A^R(b \xi_2)) ,
\]
where scalar products have to be taken in the appropriate spaces \( \tilde{\mathcal{H}}^k \). Altogether, the asserted right \( \mathcal{A} \)-linearity follows. Therefore the map
\[
\varepsilon_A \otimes R^A_B \otimes E^B \longrightarrow e^\text{ad}_A \otimes R^A_B \otimes E^B
\]
is well-defined and has the desired tensorial properties.

The definition of Ricci curvature involves another operation which we require to be similarly well-behaved: The orthogonal projections \( P_{\delta K} \) on \( \tilde{\mathcal{H}}^k \), see eq. (5.9), satisfy
\[
P_{\delta K} (axb) = aP_{\delta K} (xb)
\]
for all \( a, b \in \mathcal{A} \) and all \( x \in \tilde{\mathcal{H}}^k \). For a proof see [18]. This shows that projecting onto the "2–form part" of \( R^A_B \) is an \( \mathcal{A} \)-bi-module map, i.e., we may apply
\[
e^\text{ad}_A \otimes R^A_B \otimes E^B \longrightarrow e^\text{ad}_A \otimes (R^A_B)^{\perp} \otimes E^B
\]
with \( (R^A_B)^{\perp} = (1 - P_{\delta K}) R^A_B \) as in eq. (5.10).

Altogether, we arrive at the following definition of the Ricci curvature,
\[
\text{Ric} (\nabla) = e^\text{ad}_A \left( (R^A_B)^{\perp} \right) \otimes E^B \in \tilde{\mathcal{H}}^1 \otimes_\mathcal{A} \tilde{\Omega}^1_\Omega(D) ,
\]
which turns out to be independent of any choices. In the following, we will also use the abbreviation
\[
\text{Ric}_B := e^\text{ad}_A \left( (R^A_B)^{\perp} \right)
\]
for the components (which, again, are not uniquely defined).

From the components \( \text{Ric}_B \) we can pass to scalar curvature. Again, we have to make sure that all maps occurring in this process are \( \mathcal{A} \)-equivariant so as to obtain an invariant definition. For any 1-form \( \omega \in \tilde{\Omega}^1_\Omega(D) \), right-multiplication on \( \tilde{\mathcal{H}}^0 \) with \( \omega \) defines a bounded operator \( m_R(\omega) : \tilde{\mathcal{H}}^0 \longrightarrow \tilde{\mathcal{H}}^1 \), and we denote by
\[
\omega^\text{ad}_R : \tilde{\mathcal{H}}^1 \longrightarrow \tilde{\mathcal{H}}^0
\]
the adjoint of this operator. In a similar fashion as above, one establishes that
\[
(\omega a)^{\text{ad}}_R (x) = \omega^\text{ad}_R (xa^*)
\]
for all \( x \in \tilde{\mathcal{H}}^1 \) and \( a \in \mathcal{A} \). This makes it possible to define the scalar curvature \( r(\nabla) \) of a connection \( \nabla \) as
\[
r(\nabla) = (E^{B*})^{\text{ad}}_R (\text{Ric}_B) \in \tilde{\mathcal{H}}^0 .
\]
As was the case for the Ricci tensor, acting with the adjoint of $m_R(E^B)$ serves as a substitute for the “contraction of indices”. We summarize our results in the following

**Definition I.** Let $\nabla$ be a connection on the cotangent bundle $\tilde{\Omega}^1_D(\mathcal{A})$ over a non-commutative space $(\mathcal{A}, \mathcal{H}, D, \gamma)$ satisfying Assumption B. The **Riemannian curvature** $R(\nabla)$ is the left $A$–linear map

$$R(\nabla) = -\nabla^2 : \tilde{\Omega}^1_D(\mathcal{A}) \rightarrow \tilde{\Omega}^2_D(\mathcal{A}) \otimes_A \tilde{\Omega}^1_D(\mathcal{A}).$$

(5.18)

Choosing a set of generators $E^A$ of $\tilde{\Omega}^1_D(\mathcal{A})$ and dual generators $\varepsilon_A$ of $\tilde{\Omega}^1_D(\mathcal{A})^\ast$, and writing $R(\nabla) = \varepsilon_A \otimes R^A_B \otimes E^B$ as above, the **Ricci tensor** $\text{Ric}(\nabla)$ is given by

$$\text{Ric}(\nabla) = \text{Ric}_B \otimes E^B \in \tilde{\mathcal{H}}^1 \otimes_A \Omega^1_D(\mathcal{A}),$$

(5.19)

where $\text{Ric}_B := e^{ad}(R^A_B)^\perp$, see eqs. (5.10) and (5.16). Finally, the **scalar curvature** $r(\nabla)$ of the connection $\nabla$ is defined as

$$r(\nabla) = (E^B)^{ad}_R(\text{Ric}_B) \in \tilde{\mathcal{H}}^0,$$

(5.20)

with the notation of eq. (5.17). The tensors $\text{Ric}(\nabla)$ and $r(\nabla)$ do not depend on the choice of generators.

In [18] it is shown how to derive **Cartan structure equations** for $\nabla, R(\nabla)$ and $T(\nabla)$ in NCG, in full generality. As in classical geometry, these equations are useful for explicit calculations. In [45,49] they have been exploited to study explicit examples of non-commutative spaces arising in the Connes-Lott formulation [5,50] of the standard model.

8) **Generalized Kähler non-commutative geometry and higher supersymmetry**

In this subsection we return to basics. Recall that a spin$^c$ non-commutative space is described by some $N = 1$ supersymmetric quantum theory, formulated in terms of spectral data $(\mathcal{A}, \mathcal{H}, D, \gamma)$ with the properties specified in Definition A of subsection 1).

As in Sect. 4.1, we may ask what it is that characterizes $(\mathcal{A}, \mathcal{H}, D, \gamma)$ as the analogue of a non-commutative Kähler (or Hyperkähler, etc.) space. For a classical Kähler manifold $M$ of even real dimension $n$, the Kähler form enables one to define two Dirac operators, $D_1 := D$ and $D_2$, on the space of square-integrable sections of the bundle $S$ of Pauli-Dirac spinors, which satisfy $D_1^2 = D_2^2$ and anti-commute with each other. (Likewise, one finds two Dirac operators, $\bar{D}_1$ and $\bar{D}_2$, on the space $\mathcal{H}_p$ of charge-conjugate Pauli-Dirac spinors which anti-commute and are transformed into each other by the Kähler form.) There is an isomorphism from $S$ to $\bigoplus_{p=0}^n \Lambda^p(\tilde{\mathcal{M}})$, the bundle of holomorphic forms, under which $D_1$ is mapped to $\partial + \partial^\ast$ and $D_2$ is mapped to $i(\partial - \partial^\ast)$. Hence $\partial$ corresponds to $D_1 - iD_2$ and $\partial^\ast$ to $D_1 + iD_2$. Since $\{D_1, D_2\} = 0$ and $D_1^2 = D_2^2$, the operators $D_1 \pm iD_2$ are indeed nilpotent.

Apparently, Kähler geometry can be characterized by the existence of two supersymmetry generators $D_1$ and $D_2$ on $\mathcal{H}$, which anti-commute with each other and with the $\mathbb{Z}_2$-grading $\sigma$ (see Sect. 4.1) and whose squares are equal to each other. In quantum mechanics

$$H = D_1^2 = D_2^2$$

(5.21)
is interpreted as the Hamiltonian (generator of the time evolution) of the system.

The structure described here can be extended to non-commutative geometry in a straightforward way: Consider $N = 1$ spectral data $(A, \mathcal{H}, D, \gamma)$, as described in Definition A of subsection 1). We propose to explore the consequences of the assumption that, besides $D_1 := D$, the operator $H := D^2$ has further self-adjoint square roots, $D_2, \ldots, D_n$, such that

$$\{\gamma, D_i\} = 0, \quad \{D_i, D_j\} = 2 \delta_{ij} H,$$

for all $i, j = 1, \ldots, n$. Obviously, $H$ is a central element of the algebra generated by $D_1, \ldots, D_n$. Thus, on every eigenspace of $H$ corresponding to a non-zero eigenvalue $e \neq 0$, the operators $i D_1, \ldots, i D_n$ define a representation of the Clifford algebra over $\mathbb{R}^n$, while for $e = 0$ we have $D_1 = \ldots = D_n = 0$ (as follows from (5.22)). If $n$ is odd the existence of $\gamma$ implies that this Clifford representation is necessarily reducible. In classical geometry, one has that $n = 1$, or $n$ even.

The group of $^\ast$–automorphisms of (5.22) is $\text{SO}(n)$. There is a unitary representation $\rho$ of the group $\text{Spin}(n)$ on $\mathcal{H}$ such that

$$\rho(g) \frac{D}{\xi} \rho(g^{-1}) = \frac{D}{R(g) \xi},$$

for all $g \in \text{Spin}(n)$, where $\xi \in \mathbb{R}^n$, $\frac{D}{\xi} = \sum_{j=1}^n D_j \xi^j$, and $R : g \in \text{Spin}(n) \mapsto R(g) \in \text{SO}(n)$ is the canonical homomorphism from $\text{Spin}(n)$ to $\text{SO}(n)$. Assuming that, for $n \geq 2$, the $\mathbb{Z}_2$–grading $\gamma$ belongs to $\rho(\text{Spin}(n))$, we conclude that $n$ must be even.

In spin geometry of classical manifolds, $\rho$ commutes with the action of $A$ on $\mathcal{H}$. We say that the spectral data

$$(A, \mathcal{H}, D_1, \ldots, D_n, \gamma)$$

exhibit $N = \bar{n}$ supersymmetry iff the representation $\rho$ of $\text{Spin}(n)$ on $\mathcal{H}$ commutes with the representation of $A$ on $\mathcal{H}$.

As an example, we consider $N = \bar{2}$ supersymmetric spectral data $(A, \mathcal{H}, D_1, D_2, \gamma)$. We define

$$\partial = D_1 - i D_2, \quad \partial^* = D_1 + i D_2.$$

By (5.22),

$$\partial^2 = (\partial^*)^2 = 0 \quad \text{and} \quad \{\partial, \partial^*\} = 4H.$$

If $\mathcal{T}$ denotes the generator of the representation $\rho$ of $\text{spin}(2) \cong \mathbb{R}$ on $\mathcal{H}$ then, for a suitable normalization of $\mathcal{T}$,

$$[\mathcal{T}, \partial] = 0, \quad [\mathcal{T}, \partial^*] = -\partial^*.$$

The eigenvalues of $\mathcal{T}$ (which are, in general, neither integer nor half-integer) thus correspond to “degrees” of “holomorphic forms”. Thanks to (5.25), (5.26), the Hilbert space $\mathcal{H}$ can be interpreted as a direct sum of $\mathbb{Z}$–graded complexes for $\partial$.

This structure reminds us of classical Kähler geometry, with

$$\mathcal{A} = C(M), \quad \mathcal{H} = L^2\left(\Lambda^{\ast, 0}(M), d\text{vol}_{g}\right),$$

$$D_1 = \partial + \partial^*, \quad D_2 = i (\partial - \partial^*),$$

$\mathcal{T}$ counts the degree of a holomorphic form, and $\gamma = (-1)^{\mathcal{T}}$.

All this suggests to say that $N = \bar{2}$ supersymmetric spectral data describe “non-commutative Kähler spaces”. Similarly, one may view $N = \bar{4}$ supersymmetric spectral
data as encoding the geometry of “non-commutative Hyperkähler spaces”. The non-commutative torus, see [5,51,52] and Section 6, is an example of a non-commutative Kähler space.

9) Aspects of the algebraic topology of $N = n$ supersymmetric spectral data.

An obvious topological invariant is the index of $D$, see [15,22,17]:

$$\text{Ind} (D) = \text{tr} \left( \gamma e^{-\beta H} \right),$$

(5.27)

where $H = D^2$; compare to eq. (4.24).

With the help of the functional $\text{tr} (\gamma e^{-\beta H} (\cdot))$ one is able to construct an analogue of the Chern character and cyclic cocycles (e.g., the JLO cocycles) for the algebra $A$; see [5,52].

For spectral data with $N = \bar{n}$ supersymmetry and $n \geq 2$, one can construct an abstract analogue of Dolbeault-Hodge theory (see subsections 5.2, 7), 9)), whose precise relationship to cyclic cohomology remains to be elucidated.

5.2 Non-commutative Riemannian geometry

A notion conspicuously absent from our discussion in Sect. 5.1 is that of reality. The structure introduced there does not enable us to construct combinations of Dirac operators that are real operators. Of course, the bundle of Pauli-Dirac spinors is a complex Hermitian vector bundle, and the Dirac operator $D_A$ of Sect. 4.1 is not, in general, a real operator. However, the bundle of differential forms is a real vector bundle, and exterior differentiation and its adjoint are real operators. Our discussion of non-relativistic positronium in Sect. 4.1 suggests a way to introduce a notion of reality: To $N = \bar{n}$ spectral data $(A, \mathcal{H}, \{D_i\}_{i=1}^n, \gamma)$, one tries to associate “charge-conjugate” data $(A, \bar{\mathcal{H}}, \{\bar{D}_i\}_{i=1}^n, \bar{\gamma})$ (this corresponds to replacing the electron by the positron, as in Sect. 4.1) and then to construct a “tensor product” of these data (corresponding to positronium, see Sect. 4.1), yielding “real” spectral data (corresponding to the fact that the electric charge of positronium is zero). The definition of “charge conjugation”, for the example of classical spinc manifolds, can be inferred from many text books on quantum field theory; e.g. [53]. When $A$ is a non-commutative *–algebra a proper definition of charge conjugation requires some care [54], because one now must distinguish between left and right $A$–modules. The Hilbert space of the tensor product theory is then given by $\mathcal{H} \otimes_A \bar{\mathcal{H}}$. The details of this construction are described in [54,18] and in subsection 5) below.

In classical geometry, there are of course manifolds that do not admit any spinc structure and where one has to proceed along a different route, leading towards Riemannian geometry. In analogy, we will in this section describe supersymmetric spectral data which directly provide a notion of non-commutative Riemannian geometry.

51
1) \( N = (1, 1) \) supersymmetry and Riemannian geometry

An appropriate definition of \( N = (1, 1) \) supersymmetry can be inferred from our discussion of positronium in Sect. 4.1.

**Definition A.** The data \((A, \mathcal{H}, D, \gamma, \bar{D}, \bar{\gamma})\) are called \( N = (1, 1) \) (supersymmetric) spectral data iff

1. \( \mathcal{H} \) is a separable Hilbert space;
2. \( A \) is a unital \(*\)-algebra faithfully represented on \( \mathcal{H} \) by bounded operators;
3. \( D \) and \( \bar{D} \) are operators that are essentially self-adjoint on a common dense domain in \( \mathcal{H} \) and such that
   - \( \{D, \bar{D}\} = 0 \), \( D^2 = \bar{D}^2 = H \);
   - for each \( a \in A \), the commutators \([D, a]\) and \([\bar{D}, a]\) extend to bounded operators on \( \mathcal{H} \);
   - \( \exp(-\varepsilon H) \) is trace class for arbitrary \( \varepsilon > 0 \);
4. \( \gamma \) and \( \bar{\gamma} \) are \( \mathbb{Z}_2 \)-gradings on \( \mathcal{H} \) such that
   - \( [\gamma, a] = [\bar{\gamma}, a] = 0 \), for all \( a \in A \),
   - \( \{\gamma, D\} = [\bar{\gamma}, D] = 0 \), \( \{\gamma, \bar{D}\} = [\bar{\gamma}, \bar{D}] = 0 \).

**Remarks.**

(a) As for \( N = 1 \) supersymmetric spectral data, \( \mathbb{Z}_2 \)-gradings \( \gamma \) and \( \bar{\gamma} \) may always be introduced “by hand” if not given at the beginning:

\[
\begin{align*}
\mathcal{H} &\longrightarrow \mathcal{H} \otimes \mathbb{C}^2 \otimes \mathbb{C}^2, \\
D &\longrightarrow D \otimes \tau_1 \otimes 1, \\
\gamma &\equiv 1 \otimes \tau_3 \otimes 1, \\
\bar{\gamma} &\equiv 1 \otimes 1 \otimes \tau_3,
\end{align*}
\]

where \( \tau_1, \tau_2 \) and \( \tau_3 \) denote the usual Pauli matrices.

(b) Setting

\[
d = D - i\bar{D}, \quad d^* = D + i\bar{D},
\]

the relations in point (3) (i) of the Definition imply that

\[
d^2 = (d^*)^2 = 0, \quad dd^* + d^*d = 4H.
\]

Thus \( d \) plays the role of exterior differentiation.

(c) Setting \( \tilde{\gamma} = \gamma \bar{\gamma} \), \( D_1 := D, D_2 := \bar{D} \), the data \((A, \mathcal{H}, D_1, D_2, \tilde{\gamma})\) define \( N = 2 \) spectral data. Let \( T \) denote the generator of the representation \( \rho \) of \( \text{spin}(2) \cong \mathbb{R} \) on \( \mathcal{H} \) that implements the group \( \text{SO}(2) \cong U(1) \) of \(*\)-automorphisms of the Clifford algebra generated by \( D_1 \) and \( D_2 \), as discussed in subsection 8) of Sect. 5.1. Then

\[
[T, d] = d, \quad [T, d^*] = -d^*.
\]

Thus \( T \) counts the “degree of differential forms”.

*If*

\[
[T, a] = 0 \quad \text{for all} \ a \in A,
\]

52
we say that the data \((A, \mathcal{H}, D, \gamma, \bar{D}, \bar{\gamma}, T)\) exhibit \(N = (1, 1)\) supersymmetry.

(d) A Hodge \(*\) operator can be defined by setting \(* := \gamma\). Then we find that
\[
* d = -d^* , \quad [*, a] = 0, \quad \text{for all } a \in A .
\] (5.30)

(Alternatively, one could choose \(* := \bar{\gamma}\), with \(* d = d^* \).) On a classical manifold \(M\), a Hodge operator with the above properties exists whenever \(M\) is compact, orientable, and of even dimension. For a slightly more general definition of \(*\), applicable e.g. when \(M\) is odd-dimensional, see [18].

In conclusion, \(N = (1, 1)\) spectral data can also be described in terms of \((A, \mathcal{H}, d, \tilde{\gamma}, *, \ldots)\), with properties as in (b) – (d) above, and we call them \(N = (1, 1)\) spectral data if the operator \(T\) from remark (c) commutes with \(A\).

We now start to explore the mathematical structure described by \(N = (1, 1)\) (super-symmetric) spectral data.

2) Differential forms

Recall that the \(N = (\cdot, \cdot)\) spectral data discussed in Sect. 5.1 do not enable one to introduce any notion of reality, or, equivalently, of complex conjugation. We must show that, starting from \(N = (1, 1)\) data, one can introduce a complex conjugation with the property that \(d\) is a real operator.

We first introduce an involution \(\natural\), called complex conjugation, on the universal graded, differential algebra \(\Omega^* (A)\) of forms defined in subsection 2) of Sect. 5.1 (without the assumption that \((\delta a)^* = -\delta a\) !):
\[
\natural : \Omega^* (A) \longrightarrow \Omega^* (A)
\]
is the unique \(\mathbb{C}\)-anti-linear anti-automorphism such that
\[
\natural (a) \equiv a^\natural := a^* , \quad \natural (\delta a) \equiv (\delta a)^\natural := \delta (a^*) \quad \text{for all } a \in A .
\] (5.31)

If we write \(\hat{\gamma}\) for the mod 2 reduction of the canonical \(\mathbb{Z}\)-grading on \(\Omega^* (A)\), we have
\[
\delta \hat{\gamma} = \hat{\delta} .
\] (5.32)

We define a representation of \(\Omega^* (A)\) on \(\mathcal{H}\), again denoted by \(\pi\), by setting
\[
\pi (a) := a , \quad \pi (\delta a) := [d, a]
\] (5.33)
for all \(a \in A\). The map \(\pi\) is a \(\mathbb{Z}_2\)-graded representation in the sense that
\[
\pi (\gamma \omega \hat{\gamma}) = \gamma \pi (\omega) \gamma
\] for all \(\omega \in \Omega^* (A)\).

Although the abstract algebra of universal forms is the same as in the \(N = 1\) setting, the interpretation of the universal differential \(\delta\) has changed: In the \(N = (1, 1)\) framework, it is represented on \(\mathcal{H}\) by the nilpotent operator \(d\), instead of the self-adjoint Dirac operator \(D\), as in Sect. 5.1. This implies that
\[
\pi (\delta \omega) = [d, \pi (\omega)]_g
\] (5.34)

53
for all \( \omega \in \Omega^\bullet(\mathcal{A}) \), where \([\cdot, \cdot]_g\) denotes the graded commutator (defined with the \( \mathbb{Z}_2 \)-grading on \( \pi(\Omega^\bullet(\mathcal{A})) \)) from above. The validity of eq. (5.34) is the main difference between the \( N = (1, 1) \) and the \( N = 1 \) formalism. It ensures that there are no forms \( \omega \in \Omega^p(\mathcal{A}) \) with \( \pi(\omega) = 0 \) but \( \pi(\delta \omega) \neq 0 \).

**Proposition A.** [18] The graded vector space

\[
J = \bigoplus_{k=0}^{\infty} J^k, \quad J^k := \ker \pi \bigg|_{\Omega^k(\mathcal{A})}
\]

with \( \pi \) defined in (5.33) is a two-sided, graded, differential \( z \)-ideal of \( \Omega^\bullet(\mathcal{A}) \). As a consequence of this proposition, the algebra of differential forms

\[
\Omega^\bullet_d(\mathcal{A}) := \bigoplus_{k=0}^{\infty} \Omega^k_d(\mathcal{A}), \quad \Omega^k_d(\mathcal{A}) := \Omega^k(\mathcal{A})/J^k,
\]

is represented on the Hilbert space \( \mathcal{H} \) via \( \pi \). For later purposes, we will also need an involution on \( \Omega^\bullet_d(\mathcal{A}) \), and, according to Proposition A, it is given by the anti-linear map \( \natural \) of (5.31). Note that the “natural” involution \( \omega \mapsto \omega^* \), which is inherited from \( \mathcal{H} \) and was used in the \( N = 1 \) case, is no longer available here: The space \( \pi(\Omega^k(\mathcal{A})) \) is not closed under taking adjoints, simply because \( d \) is not self-adjoint. However, it is closed under complex conjugation \( \natural \), which is implemented on \( \mathcal{H} \) by

\[
\pi(z \omega) = \ast \pi(\omega)^\ast, \quad \omega \in \Omega^\bullet_d(\mathcal{A}),
\]

where \( \ast \) is the Hodge operator.

3) **Integration**

The integration theory follows the same lines as in the \( N = 1 \) case. The state \( f \) is given as in Definition B of Sect. 5.1, with \( 4H = 4D^2 \) written as \( \Delta = dd^\ast + d^\ast d \), see eq. (5.28). Again, we require Assumption A of subsection 5.1, 3) about the cyclicity of the integral. This yields a sesqui-linear form on \( \Omega^\bullet_d(\mathcal{A}) \) as before:

\[
(\omega, \eta) = \int \pi(\omega) \pi(\eta)^\ast
\]

for all \( \omega, \eta \in \Omega^\bullet_d(\mathcal{A}) \).

Because of the presence of the Hodge \( \ast \)-operator, the form \( (\cdot, \cdot) \) has an additional feature in the \( N = (1, 1) \) setting, namely the inner product defined in eq. (5.37) behaves like a real functional with respect to the involution \( \natural \): For \( \omega, \eta \in \Omega^\bullet_d(\mathcal{A}) \) we have that

\[
(\omega^\natural, \eta^\natural) = (\overline{\omega}, \overline{\eta})
\]

where the bar denotes ordinary complex conjugation. This is proven in [18].

Note that, in examples, \( p \)- and \( q \)-forms for \( p \neq q \) are often orthogonal w.r.t. the inner product \( (\cdot, \cdot) \). (This also implies eq. (5.38).)

Since \( \Omega^\bullet_d(\mathcal{A}) \) is a \( z \)-algebra, the statement that the ideal \( K \) defined in (5.5) is a two-sided, graded \( \ast \)-ideal of \( \Omega^\bullet(\mathcal{A}) \) is replaced by

54
Proposition B. [18] The graded kernel $K$, see eq. (5.5), of the sesqui-linear form $(\cdot, \cdot)$ is a two-sided, graded $\natural$-ideal of $\Omega^\bullet_\delta (\mathcal{A})$.

The remainder of subsection 5.1, 3) carries over to the $N = (1, 1)$ case, with the only differences that $\hat{\Omega}^\bullet (\mathcal{A})$ is a $\natural$-algebra and that the quotients $\Omega^k (\mathcal{A})/(K^k + \delta K^{k-1}) \cong \hat{\Omega}^k (\mathcal{A})/\delta K^{k-1}$ are denoted by $\hat{\Omega}^k_\delta (\mathcal{A})$.

Upon passing from $\Omega^\bullet d (\mathcal{A})$ to the algebra of square-integrable forms $\tilde{\Omega}^\bullet d (\mathcal{A})$, one might, however, lose the advantage of working with differential ideals: Whereas $J$ has this property in the $N = (1, 1)$ setting, there may exist $\omega \in K^{k-1}$ with $\delta \omega \notin K^k$. But it turns out that $K$ vanishes in many interesting examples, and, for these, we have a representation of the algebra $\hat{\Omega}^\bullet d (\mathcal{A})$ of square-integrable forms on $\tilde{\mathcal{H}}^\bullet$.

4) Unitary connections and scalar curvature

Except for the notions of unitary connections and scalar curvature, all definitions and results of subsections 5.1, 4–8) literally carry over to the $N = (1, 1)$ case. The two exceptions explicitly involve the $\natural$-involution on the algebra of differential forms, which is no longer available now. Therefore, we have to modify the definitions for $N = (1, 1)$ non-commutative geometry as follows:

Definition B. A connection $\nabla$ on a Hermitian vector bundle $(\mathcal{E}, \langle \cdot, \cdot \rangle)$ over an $N = (1, 1)$ non-commutative space is called unitary if

$$[d, \langle s, t \rangle] = \langle \nabla s, t \rangle + \langle s, \nabla t \rangle$$

for all $s, t \in \mathcal{E}$; since in general $\langle s, t \rangle \in \mathcal{A}''$, this equality is taken on the Hilbert space. The Hermitian structure on the r.s. is extended to $\mathcal{E}$-valued differential forms by

$$\langle \omega \otimes s, t \rangle = \omega \langle s, t \rangle, \quad \langle s, \eta \otimes t \rangle = \langle s, t \rangle \eta^\natural$$

for all $\omega, \eta \in \tilde{\Omega}^\bullet d (\mathcal{A})$ and $s, t \in \mathcal{E}$.

Definition C. The scalar curvature of a connection $\nabla$ on $\Omega^1_\delta (\mathcal{A})$ is defined by

$$r(\nabla) = (E^{\mathcal{B}_2})^\text{ad}_R (\text{Ric}_B) \in \tilde{\mathcal{H}}^0. \quad (5.39)$$

5) Remarks on the relation between $N = 1$ and $N = (1, 1)$ spectral data

The definitions of $N = 1$ and $N = (1, 1)$ non-commutative spectral data provide two different generalizations of classical Riemannian differential geometry. In classical geometry, one can always find an $N = (1, 1)$ description of a manifold originally given by an $N = 1$ set of data, whereas a non-commutative $N = (1, 1)$ set of spectral data appears to define a different mathematical structure than a spectral triple, because of the additional generalized Dirac operator which must be given on the Hilbert space. Thus, it is a natural and important question under which conditions on an $N = 1$ spectral triple $(\mathcal{A}, \mathcal{H}, D)$ there exists an associated $N = (1, 1)$ set of data $(\mathcal{A}, \mathcal{H}, d, \natural)$ over the same non-commutative space $\mathcal{A}$.

We have not been able, yet, to answer the question of how to pass from $N = 1$ to $N = (1, 1)$ data in full generality; but in the following we propose one construction. Our guideline is the classical case, where the main step in passing from $N = 1$ to $N = (1, 1)$
data is to replace the Hilbert space $\mathcal{H} = L^2(S)$ by $\tilde{\mathcal{H}} = L^2(\tilde{S}) \otimes_A L^2(S)$ carrying two actions of the Clifford algebra and therefore two anti-commuting Dirac operators $\mathcal{D}$ and $\bar{\mathcal{D}}$, with all the properties required in Definition A of subsection 1).

It is plausible that there are other approaches to this question, in particular ones of a more operator algebraic nature, e.g. using a “Kasparov product of spectral triples”, but we will not enter these matters here.

The first problem one meets when trying to copy the classical step from $N = 1$ to $N = (1,1)$ is that $\mathcal{H}$ should be an $A$–bi-module. To ensure this, we require that the set of $N = 1$ (even) spectral data $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is endowed with a real structure [54], i.e., that there exists an anti-unitary operator $J$ on $\mathcal{H}$ such that

$$J^2 = \epsilon \mathbf{1}, \quad J\gamma = \epsilon' \gamma J, \quad JD = DJ$$

for some (independent) signs $\epsilon, \epsilon' = \pm 1$, and such that, in addition,

$$Ja.J^* \text{ commutes with } b \text{ and } [D, b] \text{ for all } a, b \in \mathcal{A}.$$ 

This definition of a real structure was introduced by Connes in [54]; $J$ is of course related to charge conjugation, which, in this context, can be expressed in terms of Tomita’s modular conjugation (see subsection 6) below).

In the present context, $J$ provides a canonical right $A$–module structure on $\mathcal{H}$ by defining

$$\xi \cdot a := J a^* J^* \xi$$

for all $a \in \mathcal{A}, \xi \in \mathcal{H}$, see [54]. We can extend this to a right action of $\Omega^1_D(\mathcal{A})$ on $\mathcal{H}$ if we set

$$\xi \cdot \omega := J \omega^* J^* \xi$$

for all $\omega \in \Omega^1_D(\mathcal{A})$ and $\xi \in \mathcal{H}$; for simplicity, the representation symbol $\pi$ has been omitted. Note that, by the assumptions on $J$, the right action commutes with the left action of $\mathcal{A}$. Thus $\mathcal{H}$ is an $A$–bi-module. Moreover, we can form tensor products of bi-modules over the algebra $\mathcal{A}$ just as in the classical case. If $\mathcal{H}$ carries a Hermitian structure as in Definition D of Sect. 5.1, then $\mathcal{H} \otimes_A \mathcal{H}$ is endowed with a natural scalar product.

The real structure $J$ allows us to define an anti-linear “flip” operator

$$\Psi : \bigg\{ \begin{array}{l} \Omega^1_D(\mathcal{A}) \otimes_A \mathcal{H} \longrightarrow \mathcal{H} \otimes_A \Omega^1_D(\mathcal{A}) \\ \omega \otimes \xi \longmapsto J\xi \otimes \omega^* \end{array} \bigg\}.$$ 

It is straightforward to verify that $\Psi$ is well-defined and that it satisfies

$$\Psi(a s) = \Psi(s) a^*$$

for all $a \in \mathcal{A}, \ s \in \Omega^1_D(\mathcal{A}) \otimes_A \mathcal{H}$.

From now on, we furthermore assume that $\mathcal{H}$ is a projective left $\mathcal{A}$-module. (In fact, the existence of a dense projective left $\mathcal{A}$-module $\mathcal{H}_0$ inside $\mathcal{H}$ is sufficient for our purposes.) Then $\mathcal{H}$ can be equipped with connections

$$\nabla : \mathcal{H} \longrightarrow \Omega^1_D(\mathcal{A}) \otimes_A \mathcal{H},$$

i.e., $\mathbb{C}$–linear maps such that

$$\nabla (a \xi) = \delta a \otimes \xi + a \nabla \xi$$

56
for all $a \in A$ and $\xi \in \mathcal{H}$. For each connection $\nabla$ on $\mathcal{H}$, there is an "associated right-connection" $\tilde{\nabla}$ defined with the help of the flip $\Psi$:

$$\tilde{\nabla} : \begin{cases} \mathcal{H} &\to \mathcal{H} \otimes_A \Omega_D^1(A) \\ \xi &\mapsto -\Psi (\nabla J^* \xi) \end{cases}$$

$\tilde{\nabla}$ is again $\mathbb{C}$–linear and satisfies

$$\tilde{\nabla} (\xi a) = \xi \otimes a + (\nabla\xi) a .$$

A connection $\nabla$ on $\mathcal{H}$, together with its associated right connection $\tilde{\nabla}$, induces a $\mathbb{C}$–linear "tensor product connection" $\nabla$ on $\mathcal{H} \otimes_A \mathcal{H}$ of the form

$$\nabla : \begin{cases} \mathcal{H} \otimes_A \mathcal{H} &\to \mathcal{H} \otimes_A \Omega_D^1(A) \otimes_A \mathcal{H} \\ \xi_1 \otimes \xi_2 &\mapsto \nabla\xi_1 \otimes \xi_2 + \xi_1 \otimes \nabla\xi_2 \end{cases} .$$

Because of the position of the factor $\Omega_D^1(A)$, $\nabla$ is not quite a connection in the usual sense.

In the classical case, the last ingredient needed for the definition of the two Dirac operators of an $N = (1,1)$ Dirac bundle were the two anti-commuting Clifford actions $\Gamma$ and $\Gamma$ on $\mathcal{H}$. Their obvious generalizations to the non-commutative case are the $\mathbb{C}$–linear maps

$$\Gamma : \begin{cases} \mathcal{H} \otimes_A \Omega_D^1(A) \otimes_A \mathcal{H} &\to \mathcal{H} \otimes_A \mathcal{H} \\ \xi_1 \otimes \omega \otimes \xi_2 &\mapsto \xi_1 \otimes \omega \xi_2 \end{cases} \quad (5.40)$$

and

$$\Gamma : \begin{cases} \mathcal{H} \otimes_A \Omega_D^1(A) \otimes_A \mathcal{H} &\to \mathcal{H} \otimes_A \mathcal{H} \\ \xi_1 \otimes \omega \otimes \xi_2 &\mapsto \xi_1 \omega \otimes \gamma \xi_2 \end{cases} . \quad (5.41)$$

With these, we may introduce two operators $\mathcal{D}$ and $\overline{\mathcal{D}}$ on $\mathcal{H} \otimes_A \mathcal{H}$ in analogy to the classical case:

$$\mathcal{D} := \Gamma \circ \tilde{\nabla} , \quad \overline{\mathcal{D}} := \Gamma \circ \tilde{\nabla} . \quad (5.42)$$

In order to obtain a set of $N = (1,1)$ spectral data, one has to find a connection $\nabla$ on $\mathcal{H}$ which makes the operators $\mathcal{D}$ and $\overline{\mathcal{D}}$ self-adjoint and ensures that the anti-commutation relations in point (3)(i) of Definition A in subsection 1) are satisfied.

Although we are not able, in general, to prove the existence of a connection $\nabla$ on $\mathcal{H}$ which supplies $\mathcal{D}$ and $\overline{\mathcal{D}}$ with the correct algebraic properties, the naturality of the construction presented above as well as the similarity with the procedure in Sect. 4.1 leads us to expect that this problem can be solved in many cases of interest. (See Section 6 for an example.)

6) Riemannian and spin$^c$ “manifolds” in non-commutative geometry

In this section, we want to address the following question: What is the additional structure that makes an $N = (1,1)$ non-commutative space into a non-commutative “manifold”, into a spin$^c$ “manifold”, or into a quantized phase space? There exists a definition of non-commutative manifolds in terms of $K$–homology, see [5], but in the formalism introduced in the present work it is possible to find more direct criteria. In our search for the characteristic features of non-commutative manifolds we will, as before, be guided by the
classical case and by the principle that they should be natural from the physics point of view.

Extrapolating from classical geometry, we are led to the following requirement an
\(N = (1, 1)\) space \((\mathcal{A}, \mathcal{H}, d, \gamma^\ast)\) should satisfy in order to be a “manifold”. The data must extend to a set of \(N = (1, 1)\) spectral data \((\mathcal{A}, \mathcal{H}, d, T, \ast)\) where \(T\) is a self-adjoint operator on \(\mathcal{H}\) such that

i) \([T, a] = 0\) for all \(a \in \mathcal{A}\);

ii) \([T, d] = d\);

iii) \(T\) has integral spectrum, and \(\gamma\) is the mod 2 reduction of \(T\), i.e., \(\gamma = \pm 1\) on \(\mathcal{H}_\pm\), where

\[
\mathcal{H}_\pm = \text{span} \{\xi \in \mathcal{H} | T\xi = n\xi \text{ for some } n \in \mathbb{Z}, (-1)^n = \pm 1\}.
\]

Before we can formulate other properties characteristic of non-commutative manifolds, we recall some basic facts about Tomita-Takesaki theory. Let \(\mathcal{M}\) be a von Neumann algebra acting on a separable Hilbert space \(\mathcal{H}\), and assume that \(\xi_0 \in \mathcal{H}\) is a cyclic and separating vector for \(\mathcal{M}\), i.e.,

\[
\overline{\mathcal{M}\xi_0} = \mathcal{H}
\]

and

\[
a \xi_0 = 0 \implies a = 0
\]

for any \(a \in \mathcal{M}\), respectively. Then we may define an anti-linear operator \(S_0\) on \(\mathcal{H}\) by setting

\[
S_0 a \xi_0 = a^\ast \xi_0
\]

for all \(a \in \mathcal{M}\). One can show that \(S_0\) is closable, and we denote its closure by \(S\). The polar decomposition of \(S\) is written as

\[
S = J \Delta_{\frac{1}{2}}
\]

where \(J\) is an anti-unitary involutive operator, referred to as (Tomita’s) modular conjugation, and the so-called modular operator \(\Delta\) is a positive self-adjoint operator on \(\mathcal{H}\). The fundamental result of Tomita-Takesaki theory is the following theorem:

\[
J \mathcal{M} J = \mathcal{M}', \quad \Delta^{it} \mathcal{M} \Delta^{-it} = \mathcal{M}
\]

for all \(t \in \mathbb{R}\), where \(\mathcal{M}'\) denotes the commutant of \(\mathcal{M}\) on \(\mathcal{H}\). Furthermore, the vector state \(\omega_0(\cdot) := (\xi_0, \cdot \xi_0)\) is a KMS-state for the automorphism \(\sigma_t := \text{Ad}_{\Delta^{it}}\) of \(\mathcal{M}\), i.e.,

\[
\omega_0(\sigma_t (a) b) = \omega_0(b \sigma_{t-i} (a))
\]

for all \(a, b \in \mathcal{M}\) and all real \(t\).

Let \((\mathcal{A}, \mathcal{H}, d, T, \ast)\) be a set of \(N = (1, 1)\) spectral data. We define the analogue \(\text{Cl}_D(\mathcal{A})\) of the space of sections of the Clifford bundle,

\[
\text{Cl}_D(\mathcal{A}) = \{a_0 \left[ D, a_1 \right] \cdots \left[ D, a_k \right] | k \in \mathbb{Z}_+, a_i \in \mathcal{A}\},
\]

where \(D = d + d^\ast\), and, corresponding to the second generalized Dirac operator \(\overline{D} = i(d - d^\ast)\),

\[
\text{Cl}_{\overline{D}}(\mathcal{A}) = \{a_0 \left[ \overline{D}, a_1 \right] \cdots \left[ \overline{D}, a_k \right] | k \in \mathbb{Z}_+, a_i \in \mathcal{A}\}.
\]

58
In the classical setting, $\text{Cl}_D(\mathcal{A})$ and $\text{Cl}_\overline{D}(\mathcal{A})$ operate on $\mathcal{H}$ by the two actions $\Gamma$ and $\overline{\Gamma}$, respectively; see Sect. 4.1. In the general case, we notice that, in contrast to the algebras $\Omega_D(\mathcal{A})$ and $\Omega_D(\mathcal{A})$ introduced before, $\text{Cl}_D(\mathcal{A})$ and $\text{Cl}_\overline{D}(\mathcal{A})$ form $^*$-algebras of operators on $\mathcal{H}$, but are neither $\mathbb{Z}$-graded nor differential.

We want to apply Tomita-Takesaki theory to the von Neumann algebra $\mathcal{M} := (\text{Cl}_D(\mathcal{A}))''$. Suppose there exists a vector $\xi_0 \in \mathcal{H}$ which is cyclic and separating for $\mathcal{M}$, and let $J$ be the anti-unitary conjugation associated to $\mathcal{M}$ and $\xi_0$. Suppose, moreover, that for all $a \in \mathcal{A} := J\mathcal{A}J$ the operator $[\overline{D}, a]$ uniquely extends to a bounded operator on $\mathcal{H}$. Then we can form the algebra of bounded operators $\text{Cl}_\overline{D}(\mathcal{A})$ on $\mathcal{H}$ as above. The properties $J\mathcal{A}J \subset \mathcal{A}'$ and $\{D, \overline{D}\} = 0$ imply that $\text{Cl}_D(\mathcal{A})$ and $\text{Cl}_\overline{D}(\mathcal{A})$ commute in the graded sense; to arrive at truly commuting algebras, we first decompose $\text{Cl}_\overline{D}(\mathcal{A})$ into a direct sum

$$\text{Cl}_\overline{D}(\mathcal{A}) = \text{Cl}_\overline{D}^+(\mathcal{A}) \oplus \text{Cl}_\overline{D}^-(\mathcal{A})$$

with

$$\text{Cl}_\overline{D}^+(\mathcal{A}) = \{ \omega \in \text{Cl}_\overline{D}(\mathcal{A}) \mid \gamma \omega = \pm \omega \gamma \} .$$

Then we define the “twisted algebra” $\text{Cl}_\overline{D}^+(\mathcal{A}) := \text{Cl}_\overline{D}^+(\mathcal{A}) \oplus \gamma \text{Cl}_\overline{D}(\mathcal{A})$. This algebra commutes with $\text{Cl}_D(\mathcal{A})$.

We propose the following definitions: The $N = (1,1)$ spectral data $(\mathcal{A}, \mathcal{H}, d, T, \ast)$ describe a non-commutative manifold if

$$\text{Cl}_\overline{D}(\mathcal{A}) = J \text{Cl}_D(\mathcal{A}) J .$$

Furthermore, inspired by classical geometry, we say that a non-commutative manifold $(\mathcal{A}, \mathcal{H}, d, T, \ast, \xi_0)$ is spin$^c$ if the Hilbert space factorizes as a $\text{Cl}_D(\mathcal{A}) \otimes \text{Cl}_\overline{D}(\mathcal{A})$ module in the form

$$\mathcal{H} = \mathcal{H}_D \otimes_2 \mathcal{H}_\overline{D}$$

where $\mathcal{Z}$ denotes the center of $\mathcal{M}$.

Next, we introduce a notion of “quantized phase space”. We consider a set of $N = (1,1)$ spectral data $(\mathcal{F}, \mathcal{H}, d, \gamma, \ast)$, where we now think of $\mathcal{F}$ as the algebra of “phase space functions” (i.e., of pseudo-differential operators, in the Schrödinger picture of quantum mechanics; $\mathcal{F}$ is constructed as in eq. (4.39) of Sect. 4.1), rather than of functions over configuration space. We are, therefore, not postulating the existence of a cyclic and separating vector for the algebra $\text{Cl}_D(\mathcal{F})$. Instead, for each $\beta > 0$, we define the temperature or KMS state

$$\beta \mapsto \int_\beta : \left\{ \begin{array}{c} \text{Cl}_D(\mathcal{F}) \rightarrow \mathbb{C} \\ \omega \mapsto \int_\beta \omega := \frac{\text{Tr}_\mathcal{H}(\omega e^{-\beta D^2})}{\text{Tr}_\mathcal{H}(e^{-\beta D^2})} \end{array} \right. ,$$

The $\beta$-integral $\int_\beta$ clearly is a faithful state, and through the GNS-construction we obtain a faithful representation of $\text{Cl}_D(\mathcal{F})$ on a Hilbert space $\mathcal{H}_\beta$ with a cyclic and separating vector $\xi_\beta \in \mathcal{H}_\beta$ for $\mathcal{M} = (\text{Cl}_D(\mathcal{F}))''$. Each bounded operator $A \in \mathcal{B}(\mathcal{H})$ on $\mathcal{H}$ induces a bounded operator $A_\beta$ on $\mathcal{H}_\beta$; this is easily seen by computing matrix elements of $A_\beta$,

$$\langle A_\beta x, y \rangle = \int_\beta Axy^*$$

59
for all $x,y \in \mathcal{M} \subset \mathcal{H}_\beta$, and by using the explicit form of the $\beta$–integral. We denote the modular conjugation and the modular operator on $\mathcal{H}_\beta$ by $J_\beta$ and $\triangle_\beta$, respectively, and we assume that, for each $a \in \mathcal{M}$, the commutator
\[
[\overline{\partial}, J_\beta a J_\beta] = \frac{1}{i} \frac{d}{dt} \left( (e^{i\beta \overline{\partial}})_\beta J_\beta a J_\beta (e^{-i\beta \overline{\partial}})_\beta \right) \bigg|_{t=0}
\]
defines a bounded operator on $\mathcal{H}_\beta$.

Then we can define an algebra of bounded operators $\widetilde{Cl_D}^{(J_\beta \mathcal{F})}$ on $\mathcal{H}_\beta$, which is contained in the commutant of $Cl_D(\mathcal{F})$, and we say that the $N = (1,1)$ spectral data $(\mathcal{F}, \mathcal{H}, d, \gamma, *)$ describe a quantized phase space if the following equation holds:
\[
J_\beta Cl_D(\mathcal{F}) J_\beta = \widetilde{Cl_D}^{(J_\beta \mathcal{F})} .
\]

7) Algebraic topology of $N = (1,1)$ spectral data

Let $(\mathcal{A}, \mathcal{H}, D, \gamma, \overline{D}, \overline{\gamma})$ be some $N = (1,1)$ or $N = (1,1)$ supersymmetric spectral data with all the properties (1 – 4) specified in Definition A of subsection 1). We set
\[
H := D^2, \quad \overline{\gamma} = \gamma \overline{\gamma}, \quad * = \gamma .
\]

Then we can define the Euler number $\chi$ and the Hirzebruch signature $\tau$ as in formulae (4.41) and (4.42) of Sect. 4.1:
\[
\chi = \chi (\mathcal{A}, \mathcal{H}, D, \gamma, \overline{D}, \overline{\gamma}) := \text{tr}_\mathcal{H} (\overline{\gamma} e^{-\beta H}) ,
\]
and
\[
\tau := \text{tr}_\mathcal{H} (* e^{-\beta H}) .
\]

They are independent of $\beta$ and define homotopy invariants of the spectral data.

The data $(\mathcal{A}, \mathcal{H}, D, \overline{\gamma})$ permit us to introduce a functional $\text{tr}_\mathcal{H} (\overline{\gamma} e^{-\beta H}(\cdot))$ that gives rise to a JLO cyclic cocycle (the “Euler cocycle”) for the algebra $\mathcal{A}$. Likewise, the data $(\mathcal{A}, \mathcal{H}, D, *)$ yield the functional $\text{tr}_\mathcal{H} (* e^{-\beta H}(\cdot))$ and give rise to a second JLO cyclic cocycle (the “signature cocycle”). See [5,52,55] for the construction of such cocycles.

What is, perhaps, more useful is that the $N = (1,1)$ data $(\mathcal{A}, \mathcal{H}, D, \gamma, \overline{D}, \overline{\gamma})$ give rise to a de Rham–Hodge theory on $\mathcal{H}$. In order not to get lost in somewhat uninteresting generalities, we only consider $N = (1,1)$ spectral data, but see [55] for more general results. As usual, we introduce exterior differentiation and its adjoint by setting
\[
d := D - i \overline{D} , \quad d^* = D + i \overline{D} .
\]

Furthermore, there exists a $\mathbb{Z}$–grading operator $T$ such that
\[
[T, a] = 0 , \quad [T, d] = d , \quad [T, d^*] = -d^* ;
\]
see Remarks (b) and (c) of subsection 1). Let $\mathcal{H}_0 \subseteq \mathcal{H}$ be an arbitrary (e.g. minimal, non-zero) subspace of $\mathcal{H}$ invariant under the $*$-algebra generated by $\{ \mathcal{A}, D, \gamma, \overline{D}, \overline{\gamma}, T \}$.

One can normalize $T$ such that $\text{spec} T \subseteq \mathbb{Z}$, hence $\mathcal{H}_0$ becomes a $\mathbb{Z}$–graded complex:
\[
\mathcal{H}_0 = \bigoplus_{p \in \mathbb{Z}} \mathcal{H}_0^p ,
\]

60
where $\mathcal{H}_0^p$ is the eigenspace of $T$ corresponding to the eigenvalue $p \in \mathbb{Z}$. Furthermore, $\mathcal{H}_0^p$ is invariant under $\mathcal{A}$, and (5.47) implies that

$$d : \mathcal{H}_0^p \longrightarrow \mathcal{H}_0^{p+1}, \quad d^* : \mathcal{H}_0^p \longrightarrow \mathcal{H}_0^{p-1}. \tag{5.49}$$

We say that $\mathcal{H}_0^p$ is the subspace of “vector $p$–forms” and define the $p$th cohomology space by

$$H_d^p := \ker \left( d \mid_{\mathcal{H}_0^p} \right) / \operatorname{im} \left( d \mid_{\mathcal{H}_0^{p-1}} \right). \tag{5.50}$$

A harmonic vector form $\psi \in \mathcal{H}_0$ is one that satisfies

$$d \psi = d^* \psi = 0. \tag{5.51}$$

Since $4H = dd^* + d^*d$, and $d^*$ is the adjoint of $d$ on $\mathcal{H}$, we conclude that

$$\text{if } H \psi = 0, \quad \psi \text{ is harmonic} \iff \psi \text{ is harmonic}. \tag{5.52}$$

Let $\mathcal{H}_0^p \subset \mathcal{H}_0^p$ denote the subspace of harmonic vector $p$–forms. Then the usual arguments show that

$$\mathcal{H}_0^p = \mathcal{H}_h^p \oplus d\mathcal{H}_0^{p-1} \oplus d^*\mathcal{H}_0^{p+1} \tag{5.53}$$

(Hodge decomposition) and

$$H_d^p \cong \mathcal{H}_h^p \tag{5.54}$$

as vector spaces. By (5.52),

$$\left( \ker H \right)^\perp = d\mathcal{H}_0 + d^*\mathcal{H}_0, \tag{5.55}$$

and it follows easily that the cohomology of $d$ is trivial on $\left( \ker H \right)^\perp$. In particular, if supersymmetry is spontaneously broken, in the sense [15] that $\ker H = \{0\}$, then $H_d^p = \{0\}$, for all $p$. But as the example in Sect. 4.1, below (4.50), shows, we should absolutely not jump to the conclusion that the data $(\mathcal{A}, \mathcal{H}_0, \mathcal{D}, \gamma, \bar{\mathcal{D}}, \bar{\gamma})$ describe a non-commutative space with “trivial homology”. We will expand on this issue below.

Note that from (5.53–55) we can conclude that

$$\chi_0 := \text{tr}_{\mathcal{H}_0} \left( (-1)^T e^{-\beta H} \right) = \sum_p (-1)^p B_p, \tag{5.56}$$

where $B_p = \dim \mathcal{H}_h^p = \dim H_d^p$ is the $p$th Betti number. The absolute convergence of the sum on the r.s. follows from the assumption that $e^{-\beta H}$ is trace class. Likewise, one can derive a formula for $\tau_0$.

Let us next examine the cohomology of graded commutation by $d$ on the algebra $\Omega_d^*(\mathcal{A})$ of differential forms. This task is indispensable in view of the above remark that if supersymmetry is spontaneously broken in the sense that $\ker H = \{0\}$ then the cohomology spaces $H_d^p$ are all trivial. By construction, $\Omega_d^*(\mathcal{A})$ is a unital, graded, differential $k$-algebra – see subsection 2), (5.35) and (5.36) – with a faithful $k$-representation $\pi$ on $\mathcal{H}$. In the following, we omit the symbol $\pi$. For $\alpha \in \Omega_d^*(\mathcal{A})$, we define

$$\delta \alpha := [d, \alpha]_g, \tag{5.56}$$

61
where $[\cdot, \cdot]_g$ is the $\mathbb{Z}_2$–graded commutator, and
\[
\tau \alpha := [T, \alpha]; \quad \tau \alpha = n \alpha \iff \alpha \in \Omega^n_d(\mathcal{A}). \tag{5.57}
\]
We define cohomology spaces by setting
\[
H^n_d(\mathcal{A}) := \ker \left( \delta |_{\Omega^n_d(\mathcal{A})} \right) / \text{im} \left( \delta |_{\Omega^{n-1}_d(\mathcal{A})} \right). \tag{5.58}
\]
Thanks to the graded Leibniz rule obeyed by $\delta$,
\[
H^\bullet_d(\mathcal{A}) := \ker \left( \delta |_{\Omega^\bullet_d(\mathcal{A})} \right) / \text{im} \left( \delta |_{\Omega^{\bullet-1}_d(\mathcal{A})} \right) \tag{5.59}
\]
is a unital, graded $\sharp$-algebra.

Let us suppose there is a vector $\varphi_0 \in \mathcal{H}_0$ which is cyclic and separating for $\Omega^\bullet_d(\mathcal{A})$ and which is closed, i.e., $d \varphi_0 = 0$. Then (under some hypothesis of “elliptic regularity”) one finds that
\[
H^p_d(\mathcal{A}) \cong H^p_0, \text{ for all } p. \tag{5.60}
\]
The situation described here is the one encountered in the de Rham-Hodge theory of classical, smooth, compact manifolds. But if supersymmetry is spontaneously broken (i.e., if $H$ is strictly positive) then a vector $\varphi_0$ with the properties assumed above does not exist.

From the point of view of the theory of $^*$-algebras and of quantum theory, the formalism developed so far has a drawback: The algebra $\Omega^\bullet_d(\mathcal{A})$ is not a $^*$-algebra, because $d \neq d^*$. Given $a \in \mathcal{A}$, we define
\[
\delta^* a := [a, d^*]. \tag{5.61}
\]
This gives rise to a graded $\sharp$-algebra (the algebra of “poly-vector fields”)
\[
\Omega^\bullet_d(\mathcal{A}) = (\Omega^\bullet_d(\mathcal{A}))^*,
\]
with a graded differential $\delta^*$ given by
\[
\delta^* \alpha := [\alpha, d^*]_g \tag{5.62}
\]
for all $\alpha \in \Omega^\bullet_d(\mathcal{A})$. We define $\Phi^\bullet_d(\mathcal{A})$, the “field algebra”, to be the smallest $^*$-algebra of (generally unbounded) operators generated by $\Omega^\bullet_d(\mathcal{A})$ and $\Omega^\bullet_d(\mathcal{A})$ which is closed under the action of $\delta$ and $\delta^*$. (Note that the graded commutator of $d$ with the adjoint of a differential form is, in general, an unbounded operator.) Alternatively, $\Phi^\bullet_d(\mathcal{A})$ can also be defined as the $^*$-algebra generated by $\mathcal{A}$ and by arbitrary multiple graded commutators of $\mathcal{D}$ and $\bar{\mathcal{D}}$ with elements of $\mathcal{A}$. $\Phi^\bullet_d(\mathcal{A})$ is obviously $\mathbb{Z}_2$–graded and, for $N = (1,1)$ spectral data (for which a $\mathbb{Z}$–grading operator $T$ exists), it is $\mathbb{Z}$–graded:
\[
\Phi^\bullet_d(\mathcal{A}) = \bigoplus_{n \in \mathbb{Z}} \Phi^n_d(\mathcal{A}), \tag{5.63}
\]
Operators $\delta$ and $\delta^*$ are defined on $\Phi^\bullet_d(\mathcal{A})$ as in (5.56), (5.62), and $\delta : \Phi^n_d(\mathcal{A}) \longrightarrow \Phi^{n+1}_d(\mathcal{A})$, $\delta^* : \Phi^n_d(\mathcal{A}) \longrightarrow \Phi^{n-1}_d(\mathcal{A})$, with $\delta^2 = (\delta^*)^2 = 0$. Thus $\Phi^\bullet_d(\mathcal{A})$ is a graded complex for $\delta$ (and for $\delta^*$). In the situation described above eq. (5.60), the study of the complex $(\Phi^\bullet_d(\mathcal{A}), \delta^\#)$,
\[ \delta^\# = \delta \text{ or } \delta^*, \text{ does not yield any interesting results beyond those of de Rham-Hodge theory. In general, this complex is not very well understood. It may be useful to study it in connection with notions of “diffeomorphisms of non-commutative spaces” and with deformation theory (see Sect. 5.3).} \]

There is a theory dual to the cohomology theory for the complexes \((\Omega_d^*(\mathcal{A}), \delta)\) and \((\Phi_d^*(\mathcal{A}), \delta^\#)\), see [5,52,55]. It involves the notions of currents, which are operators analogous to currents in de Rham theory. A current \(C\) is an arbitrary (densely defined, closed, ...) operator on \(\mathcal{H}\) commuting with all elements of the algebra \(\mathcal{A}\) and such that

\[ \{ \tilde{\gamma}, C \} = 0 \quad (C \text{ is odd}), \quad \text{or} \quad [\tilde{\gamma}, C] = 0 \quad (C \text{ is even}). \tag{5.64} \]

We say that \(C\) is closed iff

\[ [d, C]_g = 0. \tag{5.65} \]

Obviously, \(C = 1\) (the unit of \(\mathcal{A}\)) is a closed, even current, while \(C = \tilde{\gamma}\) is an even current which is not closed. Note that closed, even currents commute with \(\Omega_d^*(\mathcal{A})\), while closed, odd currents \textit{graded-commute} with \(\Omega_d^*(\mathcal{A})\). Given a current \(C\), we would like to study functionals (“signed weights”)

\[ \text{Tr}(\tilde{\gamma} C(\cdot)), \]

where \(\text{Tr}\) is some trace to be specified more precisely. For this purpose, one can define “regularized”, multi-linear functionals

\[ \rho^\beta_C (\alpha_1 (\tau_1), \ldots, \alpha_n (\tau_n)) := \text{tr}_\mathcal{H} (\tilde{\gamma} C e^{-\beta - \tau_1 + \tau_1} H \alpha_1 e^{(\tau_1 - \tau_2) H} \alpha_2 \ldots e^{(\tau_n - \tau_1) H} \alpha_n), \tag{5.66} \]

where \(0 \leq \tau_1 \leq \tau_2 \leq \ldots \leq \tau_n < \beta, \ \alpha_1, \ldots, \alpha_n \in \Omega_d^*(\mathcal{A})\). (or in \(\Phi_d^*(\mathcal{A})\), \(n = 0, 1, 2, \ldots\). One may then attempt to construct a weight by considering the limiting functional

\[ \int_C (\cdot) := \lim_{\beta \to 0} \rho^\beta_C (\cdot), \]

where \(\text{Res}\) is a prescription for choosing a residue of \(\rho^\beta_C (\cdot)\) when \(\beta \downarrow 0\), e.g., \(\lim_{\beta \to 0} Z^{-1}_\beta \rho^\beta_C (\cdot)\), for some function \(Z_\beta\).

Note that the ordinary integral can be written as \(\tilde{\int} (\cdot) = \int (\cdot)\), with \(Z_\beta = \text{tr}_\mathcal{H} (e^{-\beta H})\).

The functionals in (5.66) and their limits as \(\beta \downarrow 0\) are building blocks for Hochschild- and cyclic cocycles, see [5,52].

One easily verifies that if \(C\) is odd then (5.66) vanishes whenever the form \(\alpha_1 \cdots \alpha_n\) is even; while if \(C\) is even (5.66) vanishes whenever \(\alpha_1 \cdots \alpha_n\) is odd.

If \(C\) is closed then, for arbitrary \(\alpha_1\) and \(\alpha_2\),

\[ \rho^\beta_C (\delta \alpha_1 (\tau_1), \alpha_2 (\tau_2)) = (-1)^{\deg \alpha_1 + 1} \rho^\beta_C (\alpha_1 (\tau_1), \delta \alpha_2 (\tau_2)), \]

(and similarly with \(C\) replaced by \(C^*\) and \(\delta\) by \(\delta^*\), respectively); in particular, we conclude that

\[ \rho^\beta_C (\delta \alpha (\tau)) = \rho^\beta_C (\delta \alpha, 1) = \rho^\beta_C (\alpha, \delta 1) = 0. \]

Thus, weights corresponding to \textit{closed} currents \textit{vanish} on \textit{exact} forms. (If, in addition, \(C\) is self-adjoint similar identities hold when \(\delta\) is replaced by \(\delta^\# = \delta\) or \(\delta^*\).)
If $C$ is a closed current commuting with $e^{itH}$, $t \in \mathbb{R}$, then $\rho_C^\beta$ satisfies the graded KMS condition

$$\rho_C^\beta(\alpha_1, \alpha_2(\tau)) = (-1)^{\deg \alpha_1 \cdot \deg \alpha_2} \rho_C^\beta(\alpha_2, \alpha_1(\beta - \tau)).$$

Formally, this identity continues to hold for the weight $f(\cdot)$ (with $\tau = 0$, $\beta \downarrow 0$), even if $C$ does not commute with $e^{itH}$, $t \in \mathbb{R}$. It should be regarded as a characteristic property of weights corresponding to closed currents.

Ultimately, one should probably explore the topology of non-commutative (phase) spaces $(\mathcal{A}, \alpha_{\tau})$, where $\alpha_{\tau}, \tau \in \mathbb{R}$, is a $^*$-automorphism group of $\mathcal{A}$, by studying the theory of “superselection sectors” [58,59] of $(\mathcal{A}, \alpha_{\tau})$, i.e., of inequivalent irreducible representations of $\mathcal{A}$ with the property that $\alpha_{\tau}$ is implemented by a unitary group $e^{i\tau H}$, with $H \geq 0$.

8) Central extensions of supersymmetry, and equivariance

We consider spectral data $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma, \bar{\gamma}, \bar{D}, \bar{\gamma})$ with all the properties specified in Definition A of subsection 1), except that, in point (3) (i), we only assume that

$$(3) \ (i') \quad \{\mathcal{D}, \bar{\mathcal{D}}\} = 0 . \quad (5.67)$$

We define three operators

$$H = \frac{1}{2} (\mathcal{D}^2 + \bar{\mathcal{D}}^2) , \quad P \equiv P_1 := \frac{1}{2} (\mathcal{D}^2 - \bar{\mathcal{D}}^2) , \quad P_2 := \frac{i}{2} \bar{\gamma} (\mathcal{D} \bar{\mathcal{D}} - \bar{\mathcal{D}} \mathcal{D}) . \quad (5.68)$$

If $\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\tau_1 = \begin{pmatrix} 0 & \bar{\gamma} \\ \bar{\gamma} & 0 \end{pmatrix}$, $\tau_2 = \begin{pmatrix} 0 & -i\bar{\gamma} \\ i\bar{\gamma} & 0 \end{pmatrix}$, $\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\bar{\gamma} = \gamma \bar{\gamma}$, denote the “graded” Pauli matrices, and

$$\mathcal{D} := \begin{pmatrix} \mathcal{D} \\ \bar{\mathcal{D}} \end{pmatrix}$$

then

$$H = \frac{1}{2} \mathcal{D}^* \tau_0 \mathcal{D}, \quad P_1 = \frac{1}{2} \mathcal{D}^* \tau_3 \mathcal{D}, \quad P_2 = \frac{1}{2} \mathcal{D}^* \tau_2 \mathcal{D} .$$

The operator $\frac{i}{2} \mathcal{D}^* \tau_1 \mathcal{D}$ vanishes by (5.67). Note that, formally, $H, P_1$ and $P_2$ are commuting, self-adjoint operators on $\mathcal{H}$ with the property that, for every unit vector $\underline{n} \in \mathbb{R}^3$,

$$H + \underline{n} \cdot P = \frac{1}{2} \mathcal{D}^* (1 + \underline{n} \cdot \tau) \mathcal{D} \geq 0 . \quad (5.69)$$

The physicists might want to call this the “relativistic spectrum condition”. Moreover, formally, $\mathcal{D}$ and $\bar{\mathcal{D}}$ commute with $H, P$ and $P_2$. (We do not enter a discussion of the functional analysis necessary to make these formal calculations rigorous facts.) Note that $H$ and $P$ are central elements of the graded Lie algebra spanned by $\mathcal{D}, \bar{\mathcal{D}}, H$ and $P$.

Defining

$$d := \mathcal{D} - i \bar{\mathcal{D}} , \quad d^* := \mathcal{D} + i \bar{\mathcal{D}} , \quad (5.70)$$

we find that

$$\{d, d\} = \{d^*, d^*\} = 4P , \quad (5.71)$$

and

$$\{d, d^*\} = 4H . \quad (5.72)$$
If \( P \neq 0 \) we say that the relations (5.71) or (5.67) describe centrally extended \( N = (1,1) \) supersymmetry, [57].

Centrally extended \( N = (1,1) \) supersymmetry is well known to mathematicians: Let \((M,g)\) be a Riemannian manifold, and let \(X\) be a Killing vector field. If we define

\[
d = d + i \lambda X \Rightarrow a^* \circ \nabla + i \lambda a(\xi) ,
\]

where \( \xi \) is the one-form corresponding to \( X \) by \( \xi(Y) = g(X,Y) \), and \( a^* \) and \( a \) are as in Section 4, then

\[
\{d,d\} = \{d^*,d^*\} = 2i \lambda \{d,X \Rightarrow \} \equiv 2i \lambda L_X ,
\]

where \( L_X \) is the Lie derivative in the direction of \( X \). Thus the operator \( P \) in (5.71) plays the role of a Lie derivative in the direction of some Killing vector field and thus generates an action of \( S^1 \) (or \( \mathbb{R} \)) on the non-commutative space described by \((A,H,D,\gamma,\bar{D},\bar{\gamma})\).

Centrally extended supersymmetry always appears in quantum field theory, where \( H \) has an interpretation as Hamiltonian (generator of time translations), and \( P \) is the momentum operator (generator of space translations, if space is one-dimensional).

If we continue to assume that \( \exp(-\varepsilon H) \) is trace class, for arbitrary \( \varepsilon > 0 \), then the spectrum of \( P \) is discrete. If 0 is in the spectrum of \( H \) then it is also in the spectrum of \( P \), because of the “relativistic spectrum condition” (5.69), and the subspace

\[
\mathcal{H}_0 = \{ \psi \in \mathcal{H} | P\psi = 0 \}
\]

is non-empty. We may then restrict the operators \( H, D, \gamma, \bar{D} \) and \( \bar{\gamma} \) to \( \mathcal{H}_0 \), where they generate a standard \( N = (1,1) \) supersymmetry algebra of the type considered in previous sections.

Assuming for simplicity that \( P \) generates an \( S^1 \)-action and that \( e^{i\theta P}, \theta \in [0,2\pi) \), defines a \( * \)-automorphism group of \( A \) (i.e. \( e^{i\theta P} a e^{-i\theta P} \in \mathcal{A} \), for all \( a \in \mathcal{A} \)), we can define the fixed-point subalgebra

\[
\mathcal{A}_0 = \left\{ \frac{1}{2\pi} \int_0^{2\pi} d\theta \ e^{i\theta P} a e^{-i\theta P} \bigg| a \in \mathcal{A} \right\} ,
\]

and the data \((\mathcal{A}_0,\mathcal{H}_0,\mathcal{D}_0,\gamma_0,\bar{D}_0,\bar{\gamma}_0)\), where \( \mathcal{D}_0 = \mathcal{D}|_{\mathcal{H}_0} \), \( \gamma_0 = \gamma|_{\mathcal{H}_0} \), are \( N = (1,1) \) spectral data in the sense of subsection 1).

The interesting topological invariants, in the present context, are

\[
\chi = \text{tr}_{\mathcal{H}_0} (\tilde{\gamma}_0 \ e^{-\beta \mathcal{H}_0}) = \text{tr}_{\mathcal{H}} (\tilde{\gamma} \ e^{-\beta H} e^{i\theta P}) ;
\]

(5.76)

\[(\text{the r.s. is easily seen to be independent of } \theta)\]

\[
\tau(\theta) := \text{tr}_{\mathcal{H}} (\ast e^{-\beta H} e^{i\theta P}) ,
\]

(5.77)

where \( * \) is the Hodge operator introduced in Remark (d) of subsection 1); one easily checks that \( \tau(M;\theta) \) is independent of \( \beta \). For classical Riemannian manifolds, one can derive Lefschetz fixed point formulae for \( \chi \) and \( \tau(\theta) \) — as well as for the \( \hat{A} \) genus of \( N = 1 \) spectral data; see [56,22]. The easiest example is

\[
\chi(M) = \sum_i \chi(M_i) ,
\]

65
where \( M_1, M_2, \ldots \) are the connected components of the fixed point set of the Killing vector field \( X \) (Lefschetz fixed point theorem).

Of course, for the data \((A_0, H_0, D_0, \gamma_0, \bar{D}_0, \bar{\gamma}_0)\), all the results of subsection 7) can be carried over. Here we are entering the realm of \( S^1 \)-equivariant cohomology, but we shall not develop this theme here, beyond saying that the \( S^1 \)-equivariant cohomology is determined by \( H^*_d(A_0) \); see Sect. 5.3. An example that is important in the study of two-dimensional supersymmetric \( \sigma \)-models is to choose as an algebra \( \mathcal{A} \) “something like” \( C(M^S) \), where \( M^S \) is the loop space over some compact manifold \( M \) (interpreted as the target space of the \( \sigma \)-model), and \( H \) is the Hilbert space of physical state vectors of the \( \sigma \)-model. The operator \( P \) is chosen to represent the generator of rigid rotations, \( \varphi \mapsto \varphi + \theta, \ \theta \in [0, 2\pi) \), of loops in \( M^S \) on \( H \). Considering a \( \sigma \)-model exhibiting “unbroken” \( N = (1, 1) \) supersymmetry [15], one concludes, formally, that the de Rham-Hodge theory for \((A_0, H_0, D_0, \gamma_0, \bar{D}_0, \bar{\gamma}_0)\) yields the de Rham cohomology of \( M \). Upon closer examination of the situation, one finds that the natural “algebra” \( \mathcal{A} \) provided by a quantized supersymmetric \( \sigma \)-model is really an “algebra of functions” on quantum phase space over some deformation of \( M^S \). The deformation of target space may be “invisible” at the level of de Rham theory (although the algebraic structure of \( H^*_d(A_0) \) is generally not that of a graded-commutative algebra); but when one attempts to reconstruct the Riemannian geometry of target space form \((A_0, H_0, D_0, \gamma_0, \bar{D}_0, \bar{\gamma}_0)\) one may find that it is surprisingly different from that of \( M \). The example of the supersymmetric WZW model (where \( M \) is a semi-simple, compact Lie group) is instructive; see [24] and Section 7.

9) \( N = (n, n) \) supersymmetry, and supersymmetry breaking

In this section we describe non-commutative generalizations of complex Hermitian, symplectic, Kähler, hypercomplex, and Hyperkähler geometry in terms of \( N = (n, n) \) supersymmetric spectral data with partially broken supersymmetry, following the ideas of subsection 8) of Sect. 5.1.

**Definition D.** The data \((A, H, \{D_i\}_{i=1}^n, \gamma, \{\bar{D}_i\}_{i=1}^n, \bar{\gamma})\) are called \( N = (n, n) \) (supersymmetric) spectral data iff properties (1) and (2) of Definition A in subsection 1) hold, and

\[
\text{(3) } \{D_i\}_{i=1}^n, \{\bar{D}_i\}_{i=1}^n \text{ are essentially self-adjoint on a common dense domain in } H \text{ such that}
\]

\[
\begin{align*}
(i) \quad & \{D_i, D_j\} = \{\bar{D}_i, \bar{D}_j\} = 0, \text{ for all } i \neq j, \{D_i, \bar{D}_j\} = 0, \text{ for all } i, j; \\
(ii) \quad & \text{for each } a \in \mathcal{A}, \text{ the commutators } [D_i, a] \text{ and } [\bar{D}_i, a], i = 1, \ldots, n, \text{ extend to bounded operators on } H; \\
(iii) \quad & \text{defining } H := \sum_{i=1}^n(D_i^2 + \bar{D}_i^2), \text{ the operator } \exp(-\varepsilon H) \text{ is required to be trace class, for arbitrary } \varepsilon > 0; \\
\end{align*}
\]

\[
\text{(4) } \gamma \text{ and } \bar{\gamma} \text{ are } \mathbb{Z}_2-\text{gradings on } H \text{ such that}
\]

\[
\begin{align*}
(i) \quad & [\gamma, a] = [\bar{\gamma}, a] = 0, \text{ for all } a \in \mathcal{A}; \\
(ii) \quad & \{\gamma, D_i\} = [\gamma, D_i] = 0, \{\bar{\gamma}, D_i\} = [\bar{\gamma}, D_i] = 0, \text{ for all } i.
\end{align*}
\]

66
The operators
\[ D_i, \, \bar{D}_i, \, L_i := D_i^2, \, \bar{L}_i := \bar{D}_i^2, \, \quad i = 1, \ldots, n, \quad (5.78) \]
form a graded Lie algebra with \( L_i, \bar{L}_i \) as central elements. The latter operators are positive and commute with \( H \); thus they have discrete spectrum, by Definition D, (3 iii).

On each eigenspace of \( \{ L_1, \ldots, L_n, \bar{L}_1, \ldots, \bar{L}_n \} \), the generalized Dirac operators \( D_1, \ldots, D_n, \bar{D}_1, \ldots, \bar{D}_n \) form a \textit{finite-dimensional} representation of a Clifford algebra in \( m \leq 2n \) dimensions. The automorphism group of the graded Lie algebra generated by \( D_1, \ldots, D_n, \bar{D}_1, \ldots, \bar{D}_n, L_1, \ldots, L_n, \) and \( \bar{L}_1, \ldots, \bar{L}_n \) is thus unitarily implemented on \( \mathcal{H} \). If this representation commutes with \( A \) we say that the spectral data \( (A, \mathcal{H}, \{ D_i \}_{i=1}^n, \gamma, \{ \bar{D}_i \}, \bar{\gamma}) \) are \( N = (n,n) \) supersymmetric. The general theory of \( N = (n,n) \) and \( N = (n,n) \) supersymmetric data can now be developed by combining subsection 8) of Sect. 5.1 with subsections 8) and 7) of Sect. 5.2, above. Apart from a few details, there is nothing interesting to invent or to check.

We do, however, discover \textit{new} geometric structure if we study various ways of \textit{breaking} supersymmetry, see also [18]. In order to stay as close to notions in classical geometry as possible, it is useful to reformulate \( N = (n,n) \) spectral data in an alternative way. Since we shall not aim at full generality here, we restrict our attention to \( N = (n,n) \) spectral data with a “charge-conjugation symmetry”:
\[ L_i = \bar{L}_i, \quad i = 1, \ldots, n. \quad (5.79) \]
(In the general case, we pass to the subspace \( \mathcal{H}_0 \) of \( \mathcal{H} \) on which (5.79) holds; see subsection 8) above.) Thanks to (5.79) we can define \( n \) nilpotent differentials
\[ d_j := D_j - i \bar{D}_j, \quad j = 1, \ldots, n, \quad (5.80) \]
with adjoints \( d_j^* = D_j + i \bar{D}_j \). We introduce a \( \mathbb{Z}_2 \)-grading \( \tilde{\gamma} \) and a Hodge * operator by
\[ \tilde{\gamma} := \gamma \bar{\gamma}, \quad * = \gamma. \quad (5.81) \]
Then
\[ d_j = (d_j^*)^2 = 0, \quad \{ \tilde{\gamma}, d_j \} = 0, \quad * d_j * = -d_j^*, \quad (5.82) \]
for all \( j = 1, \ldots, n \).

If the spectral data have \( N = (n,n) \) supersymmetry, there is also a \( \mathbb{Z} \)-grading operator \( T \) such that for \( j = 1, \ldots, n \)
\[ [T, d_j] = d_j, \quad [T, d_j^*] = -d_j^*. \quad (5.83) \]
The reformulation (5.79–83) makes it clear that there are \textit{two} types of automorphisms of \( N = (n,n) \) spectral data, which we call “\textit{horizontal}” and “\textit{vertical}”. Let \( G \) be the subgroup of \( * \)-automorphisms of the graded Lie algebra generated by \( \{ D_i, \bar{D}_i, L_i, \bar{L}_i \}_{i=1}^n \) that is implemented on \( \mathcal{H} \) by a unitary representation \( \pi \) with the property that \( \pi(G) \) commutes with \( A \). Let \( \mathcal{G} \) denote the Lie algebra of \( G \). An element \( J \in d\pi(\mathcal{G}) \) is said to be the generator of a \textit{horizontal symmetry} iff
\[ [J, d_i] = J_i \cdot d_j, \quad (5.84) \]
for some complex numbers $J^j_i$ (we are using the summation convention). A similar equation for $d^*_i$ follows by taking the adjoint of (5.84). An element $\Omega \in d\pi(G^C)$ is said to be the generator of a \textit{vertical symmetry} iff

$$[\Omega, d^*_i] = \Omega^j_i d_j ,$$

(5.85)

for some real (or complex) numbers $\Omega^j_i$. Of course, we also demand that $J$ and $\Omega$ commute with $\tilde{\gamma}$. We assume that there exists a $\mathcal{Z}$–grading operator $T$. Recall the (graded) Jacobi identities:

$$[[A,B],C] + [[B,C],A] + [[C,A],B] = 0 ,$$
$$[[A,B],C] - \{[B,C],A\} + \{[C,A],B\} = 0 ,$$
$$\{[A,B],C\} + \{[B,C],A\} + \{[C,A],B\} = 0 .$$

(5.86)

The first identity in (5.86) and eqs. (5.83,84) imply that $[T,J]$ commutes with $A$ and with $d_j$, and, since $J$ must be anti-selfadjoint, with $d^*_j$, for all $j$. Since $i[T,J]$ belongs to $d\pi(G)$, we conclude that

$$[T,J] = 0 .$$

(5.87)

By (5.88) and (5.86),

$$[T, [\Omega, d_i]] = [\Omega, [T, d_i]] - [d_i, [T, \Omega]] = 3 [\Omega, d_i] + [Z, d_i] .$$

(5.89)

If $[Z, d_i] = 0$, for all $i$ then $Z = 0$, and it follows that $\Omega$ has degree 2 and $[\Omega, d_i]$ has degree 3.

In the following, we focus our attention on spectral data with the property that all \textit{odd} elements of the graded Lie algebra generated by $d_i, d^*_i$ and $G$ have in fact $\mathcal{Z}$–degree $\pm 1$. Then it follows from (5.89) and $Z = 0$ that

$$[\Omega, d_i] = 0 , \quad i = 1, \ldots, n .$$

(5.90)

Eqs. (5.84), (5.87) and eqs. (5.85), (5.90) and (5.88) show that, in the context of classical manifold theory, a horizontal symmetry generator $J$ corresponds to a \textit{complex structure}, while a vertical symmetry operator $\Omega$ corresponds to wedging by a \textit{symplectic 2-form}; (eq. (5.90) says that a symplectic form is \textit{closed}).

It is not hard to elucidate the algebraic structure of differentials and horizontal and vertical symmetries further by repeated use of eqs. (5.83–90); but we refrain from going into further details here. Instead, we propose to change our point of view: Rather than starting from $N = (n,n)$ spectral data, we may start from $N = (1,1)$ spectral data, as in subsection 1) of Sect. 5.2, and enrich them by horizontal and/or vertical symmetries.

Thus, let $(\mathcal{A}, \mathcal{H}, d, \tilde{\gamma}, *)$ be some $N = (1,1)$ spectral data, and let $\mathcal{G}_h$ be a Lie algebra of “horizontal symmetries” represented on $\mathcal{H}$ by anti-selfadjoint operators which commute with $\mathcal{A}, \tilde{\gamma}, T$ and $*$. Let $\{J_1, \ldots, J_{n-1}\}$ be a basis for $\mathcal{G}_h$ with the property that the operators

$$d_1 := d, \quad d_k := [J_{k-1}, d], \quad k = 2, \ldots, n ,$$

(5.91)
span a \( \mathcal{G}_h \)-module under the adjoint action. The (graded) Jacobi identity (first and second equation in (5.86)) shows that

\[
\{d_i, d_j\} = 0, \quad \text{for all } i \text{ and } j.
\]  

(5.92)

However, the structure described by \( \mathcal{G}_h \) and (5.91) does not imply that \( \{d_i, d_j^*\} = 0 \) for \( i \neq j \), as would be the case if the differentials \( d_i \) were derived from \( N = (n, n) \) spectral data as in eq. (5.80). We may now introduce Dolbeault differentials

\[
\partial_k = d + i d_k, \quad \bar{\partial}_k = d - i d_k,
\]

(5.93)

\( k = 2, \ldots, n \). Assuming that, for all \( a \in \mathcal{A} \), the operator \( \{\partial_k, [\bar{\partial}_k, a]\} \) is bounded, we can introduce a bi-graded, bi-differential algebra \( \Omega^{\bullet, \bullet}_{\partial_k, \bar{\partial}_k}(\mathcal{A}) = \bigoplus_{p,q} \Omega^{p,q}_{\partial_k, \bar{\partial}_k}(\mathcal{A}) \) in the obvious way, satisfying

\[
\Omega^p_d(\mathcal{A}) = \bigoplus_{p+q=n} \Omega^{p,q}_{\partial_k, \bar{\partial}_k}(\mathcal{A});
\]

see [18]. The elements of \( \Omega^{\bullet, \bullet}_{\partial_k, \bar{\partial}_k}(\mathcal{A}) \) are called Dolbeault forms. We can construct an integral \( f \) and a metric \( \langle \cdot, \cdot \rangle \) on \( \Omega^{\bullet, \bullet}_{\partial_k, \bar{\partial}_k}(\mathcal{A}) \) (using \( H = \frac{1}{4} (dd^* + d^*d) \)). Furthermore, one can define holomorphic vector bundles, \( \mathcal{E} \), as finitely generated, projective (left-) modules for \( \mathcal{A} \) equipped with a connection \( \nabla = \nabla^{(1,0)} + \nabla^{(0,1)} \), where \( \nabla^{(p,q)} : \mathcal{E} \longrightarrow \Omega^{p,q}_{\partial_k, \bar{\partial}_k}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \) for \( p + q = 1 \), such that

\[
\nabla^{(0,1)} \circ \nabla^{(0,1)} = 0.
\]

(5.94)

See [18] for details. Apparently, the structure we are exploring here resembles the theory of (hyper)complex Hermitian manifolds in classical geometry; see [26,27,28]. We therefore say that spectral data \( (\mathcal{A}, \mathcal{H}, d, \bar{\gamma}, *, T, \mathcal{G}_h) \), where \( \mathcal{G}_h \) is a Lie algebra of horizontal symmetries as in (5.91), define a (hyper)complex Hermitian non-commutative space. The special case, where \( \mathcal{G}_h = \mathbb{R} \), i.e., with only one complex structure \( J \), characterizes complex non-commutative geometry; the special case where \( \mathcal{G}_h = \text{su}(2) \), i.e., with three complex structures \( I, J \) and \( K \), is characteristic of standard hypercomplex geometry (see [26]). Further cases are discussed, for classical manifolds, in [27,28].

Of course, if it so happens that

\[
\{d_i, d_j^*\} = 0 \quad \text{for all } i \neq j,
\]

(5.95)

then the data \( (\mathcal{A}, \mathcal{H}, d, \bar{\gamma}, *, T, \mathcal{G}_h) \) determine \( N = (n, n) \) or \( N = (n, n) \) supersymmetric spectral data, with \( \mathcal{D}_i = d_i + d_i^* \), \( \bar{\mathcal{D}}_i = i (d_i - d_i^*) \), \( \gamma = * \) and \( \bar{\gamma} = * \bar{\gamma} \). Thus, the anticommutators

\[
\{d_i, d_j^*\}, \quad i \neq j,
\]

(5.96)

describe an explicit breaking of \( N = (n, n) \) supersymmetry, and, in geometry, broken supersymmetry is apparently a rather standard phenomenon.

Besides the algebras \( \Omega^{\bullet, \bullet}_{\partial_k, \bar{\partial}_k}(\mathcal{A}) \), we should also consider the bi-graded, differential algebras

\[
\Omega^{\bullet, \bullet}_{\partial_k, \bar{\partial}_k}(\mathcal{A}) = \bigoplus_{p,q} \Omega^{p,q}_{\partial_k, \bar{\partial}_k}(\mathcal{A}),
\]

(5.97)

which are differential algebras of (generally unbounded) operators for \( \partial_k \) and for \( \bar{\partial}_k \), but not, usually, bi-differential algebras, (unless \( \{\partial_k, \bar{\partial}_k\} = 0 \)). These algebras are defined in
an obvious way. The study of the cohomology of the complex \((\Omega_{\bar{\partial}_k,\partial_k}^{\bullet,\bullet}(A), \partial_k)\) is important in deformation theory, e.g., for the Kodaira-Spencer theory of deformations of the complex structure \(J_{k-1}\); see [60].

If, for some \(k \geq 2\), \(\{d, d_k^*\} = 0\) and if \(4L_1 = dd^* + d^*d = d_kd_k^* + d_k^*d_k = 4L_k\) i.e., if there is an unbroken \(N = (2,2)\) supersymmetry, then \(\Omega_{\partial_k,\bar{\partial}_k}^{\bullet,\bullet}(A)\) is a bi-differential algebra, and \(\Omega_{\partial_k,\bar{\partial}_k}^{\bullet,\bullet}(A), \partial_k,\bar{\partial}_k\) is a bi-complex. **Mirror symmetry** is a map from spectral data \((A, \mathcal{H}, \partial_k, T, \bar{\partial}_k, \bar{T})\) to data

\[
\left( \mathcal{B}, \mathcal{H}, \partial_k' := \partial_k, T' := T, \bar{\partial}_k' := \bar{\partial}_k, \bar{T}' := -\bar{T} \right)
\]

where \(T\) and \(\bar{T}\) are the holomorphic and anti-holomorphic \(\mathbb{Z}\)-grading operators, and \(\mathcal{B}\) is a second \(*\)-algebra on \(\mathcal{H}\) with the same properties as \(A\). (One should, perhaps, assume that the phase space algebras generated by \((A, H)\) and by \((\mathcal{B}, H)\), where \(H = L_1 = L_k\), coincide, as is true in \(N = (2,2)\) supersymmetric conformal field theory in two dimensions, see e.g. [24].)

Next, we consider \(N = (1,1)\) spectral data enriched by vertical symmetries. This leads us to a natural notion of symplectic (and “hyper-symplectic”) non-commutative geometry, [18]. Thus, let \((A, \mathcal{H}, d, \tilde{\gamma}, \ast)\) be some \(N = (1,1)\) spectral data, and let \(\Omega_1, \ldots, \Omega_n\) be a basis of vertical symmetries, in the sense that the operators

\[
d_k := [\Omega_k, d^*] \tag{5.98}
\]

are nilpotent and

\[
[T, \Omega_k] = 2\Omega_k, \quad [d, \Omega_k] = 0, \tag{5.99}
\]

for all \(k = 1, \ldots, n\). It follows from the graded Jacobi identity (second equation in (5.86)) that

\[
\{d_k, d^*\} = 0, \tag{5.100}
\]

for all \(k = 1, \ldots, n\). We are however not claiming that

\[
\{d_k, d\} = 0, \tag{5.101}
\]

because (5.101) does not follow from the structure required, so far, and is not valid in examples. Note that, from (5.98) and its adjoint, from (5.99) and from the Jacobi identity, it follows that

\[
[\Omega_l, d_k^*] = [\Omega_l, [d, \Omega_k^*]] = [[\Omega_k^*, \Omega_l], d] \tag{5.102}
\]

is an operator of degree 1 which anti-commutes with \(d\). If this operator is nilpotent then \([\Omega_k^*, \Omega_l]\) is a linear combination of \(T\) and of horizontal symmetries, i.e., is related to complex structures.

Let us consider the case where \(n = 1\), setting \(\Omega_1 := \Omega, d := d_1\) and \([\Omega, d^*] := d_2\). If \(\{d_1, d_2\} \neq 0\) then there cannot exist a horizontal symmetry relating \(d_1\) and \(d_2\). It follows that either \(i[\Omega^*, \Omega]\) is a new horizontal symmetry, or \([\Omega^*, \Omega] = T\) after appropriate normalization. The second alternative describes what one might want to call a symplectic non-commutative space. Apparently, such a space is characterized by \(N = (1,1)\) spectral data, enriched by one vertical symmetry \(\Omega\), i.e., by

\[
(A, \mathcal{H}, d_1 \equiv d, \tilde{\gamma}, \ast, \Omega) \tag{5.103}
\]

70
with the properties that
\[
d_2 := [\Omega, d^*] \text{ is nilpotent, } \quad [d, \Omega] = 0 ,
\]
\[
[T, \Omega] = 2\Omega, \quad [\Omega^*, \Omega] = T .
\] (5.104)

Obviously, the operators \( \Omega, \Omega^* \) and \( T \) determine a unitary representation of the Lie algebra \( \mathfrak{sl}_2 \), and
\[
\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} d_2^* \\ d_1^* \end{pmatrix}
\]
form two \( \mathfrak{sl}_2 \) doublets under the action of \( \Omega^*, \Omega, T \).

If \( \{d_1, d_2\} \neq 0 \) then, setting \( D_j := \frac{1}{2} (d_j + d_j^*) \), \( \bar{D}_j := \frac{1}{2} (d_j - d_j^*) \), we find that
\( (A, \mathcal{H}, D_1, D_2, \gamma = *, \bar{D}_1, \bar{D}_2, \bar{\gamma} = *\bar{\gamma}) \) are \( N = (2, 2) \) spectral data, which give rise to a non-commutative Kähler geometry.

Our analysis suggests to define a mirror map from a symplectic to a complex-Hermitian non-commutative space as a map \( m, m : (A, \mathcal{H}, d_1, d_2, \tilde{\gamma}, *) \mapsto (B, \mathcal{H}, d_1', d_2' := d_2^*, \tilde{\gamma}, *) \) (5.105)
where \( B \) is a second \( * \)-algebra on \( \mathcal{H} \) with the same properties as \( A \). The Dolbeault differentials for \( B \) are then given by
\[
\partial = d_1 - id_2^*, \quad \bar{\partial} = d_1 + id_2^* .
\] (5.106)

Of course, \( m \) cannot preserve the \( \mathbb{Z} \)-grading \( T \).

Spectral data with several vertical symmetry generators \( \Omega_1, \ldots, \Omega_n \) describe (non-commutative) geometries with several (at least \( n - 1 \)) complex structures corresponding to a possibly broken \( N = (n + 1, n + 1) \) supersymmetry.

In [18] the reader can find additional technical details on complex Hermitian and symplectic (non-commutative) geometry.

In conclusion, we have been able to characterize (non-commutative) complex-Hermitian and symplectic geometry in terms of spectral data with broken \( N = (2, 2) \) supersymmetry
\[
\{d_1, d_2^*\} \neq 0 \quad \text{or} \quad \{d_1, d_2\} \neq 0 ,
\]
respectively. Unbroken \( N = (2, 2) \) supersymmetry corresponds to (non-commutative) Kähler geometry. In spin\(^c \) geometry, as developed in Sect. 5.1, supersymmetry breaking corresponds to spectral data with Dirac operators \( D_1, D_2, \ldots \) such that
\[
\{D_1, D_2\} \neq 0 .
\]

There is some beginning of an understanding under what conditions supersymmetry can be restored by deforming the generators \( D_i, \bar{D}_i \) (or \( D_i \), resp.), \( i = 2, 3, \ldots \); see [55].

Finally, we note that one can also study asymmetric \( N = (n, m) \) or \( N = (n, m) \) supersymmetric spectral data, with \( n \neq m \), thus leaving the realm of real geometry; such data appear in superconformal field theory and string theory [29].
5.3 Reparametrization invariance, BRST cohomology, and target space supersymmetry

In this section we study non-commutative spaces described by $N = n$ or $N = (n, n)$ spectral data which admit some symmetries, called reparametrizations. If a $*$-algebra $A$ is interpreted as the “algebra of functions” over some (non-commutative) “space” then “symmetries” of this space can be described as $*$-automorphisms of $A$. They form a group denoted by $\text{Aut}(A)$. Infinitesimal symmetry transformations are $*$-derivations of $A$ and form a Lie algebra denoted by $\text{Der}(A)$. In classical geometry, where $A = C(M)$, with $M$ e.g. a smooth, compact manifold, $\text{Der}(A)$ consists of (all Lie derivatives associated with) smooth vector fields, i.e., $L \in \text{Der}(A)$ iff $L = L_X = \{d, a(\xi)\}$, where $\xi$ is the 1-form corresponding to a smooth vector field $X$ w.r.t. some Riemannian metric on $M$ (see Sect. 4.1). Thus $[d, L] = 0$ for all $L \in \text{Der}(A)$, and it follows that $L$ is automatically a derivation of the differential algebra $\Omega^{*}(A)$. (It is, however, not necessarily a derivation of $\Omega^{*+}(A)$ or of $\Phi^{*}(A)$. But this holds if $X$ is an isometry, i.e., if $L_X$ commutes with $d^{*}$.)

In non-commutative geometry, derivations of $A$ have no reason to commute with the differential $d$ of some $N = (1, 1)$ supersymmetric spectral data $(A, \mathcal{H}, d, \gamma, *)$. In general, they do not commute with $d$ and $d^{*}$ (or with $\mathcal{D}$ and $\bar{\mathcal{D}}$). One might want to call the subgroup of $\text{Aut}(A)$ commuting with $d$ the diffeomorphism group of $A$, $\text{Diff}_{d}(A)$, and the subgroup of $\text{Aut}(A)$ commuting with $d$ and with $d^{*}$ the group of isometries of $A$. Our purpose, in this section, is not primarily to study diffeomorphisms or isometries of $A$, but certain subalgebras $\mathcal{G} \subseteq \text{Der}(A)$, called algebras of infinitesimal reparametrizations of $A$, with properties described presently.

In order not to get lost in generalities, we start from $N = 1$ supersymmetric spectral data

$$(A, \mathcal{H}, D, \gamma). \quad (5.107)$$

Let $\mathcal{G}$ be some Lie subalgebra of $\text{Der}(A)$, with a basis $T_1, \ldots, T_n$. (To be on safe grounds, we temporarily assume that $\mathcal{G}$ is finite-dimensional, i.e., $n < \infty$.)

**Definition A.** $\mathcal{G}$ is a Lie algebra of infinitesimal reparametrizations of $(A, \mathcal{H}, D, \gamma)$ iff

(i) $\mathcal{G}$ is implemented on $\mathcal{H}$ in a representation $d\pi$ by anti-selfadjoint operators commuting with the $\mathbb{Z}_{2}-$grading $\gamma$; hence

$$L_j := d\pi(T_j) \quad (5.108)$$

is an anti-selfadjoint operator commuting with $\gamma$, for all $j = 1, \ldots, n$;

(ii) the graded Lie algebra $\mathcal{G}_{D, \gamma}$, generated by $d\pi(\mathcal{G})$, by $D$ and by arbitrary graded commutators thereof, satisfies

$$(\mathcal{G}_{D, \gamma})_{\text{even}} = d\pi(\mathcal{G}) \quad (5.109)$$

and is finite-dimensional,

$$\text{dim } (\mathcal{G}_{D, \gamma}) < \infty. \quad (5.110)$$

Let $L_1, \ldots, L_n$, $D_1 := D, D_2, \ldots, D_m$ be a basis of $\mathcal{G}_{D, \gamma}$, where $L_1, \ldots, L_n$ span $(\mathcal{G}_{D, \gamma})_{\text{even}}$ and $D_1, \ldots, D_m$ span $(\mathcal{G}_{D, \gamma})_{\text{odd}}$. Note that the operators $D_1, \ldots, D_m$ are self-adjoint. The structure constants of $\mathcal{G}_{D, \gamma}$

$$\{f_{ij}^{k} = -f_{ji}^{k}, g_{ij}^{k}, h_{ij}^{k} = h_{ji}^{k}\} \quad (5.111)$$
are real numbers such that
\[
\begin{align*}
[L_i, L_j] &= f^k_{ij} L_k, \\
[L_i, D_j] &= g^k_{ij} D_k, \\
\{D_i, D_j\} &= i h^k_{ij} L_k.
\end{align*}
\] (5.112)

The graded Jacobi identities (5.86) yield quadratic relations between the structure constants.

Our goal, here, is to find the \(G\)-invariant ground states of the operator
\[ H = D^2, \] (5.113)
i.e., those state vectors \(\psi \in \mathcal{H}\) that satisfy
\[ D\psi = 0, \quad L_j \psi = 0, \] (5.114)
for all \(j = 1, \ldots, n\). This problem comes up in string theory [29] and in \(M\)(embrane) theory [61].

\[ D_l \psi = 0 \] (5.115)
holds for all \(l = 1, \ldots, m\), which means that the \(G\)-invariant ground states of \(H\) are precisely the \(G_{\gamma}\)–invariant state vectors in \(\mathcal{H}\). The problem of finding these states can be viewed as a problem in BRST cohomology.

Recall that, in Sect. 4.2, the problem to find all \(G\)-invariant states in \(\mathcal{H}\) has been formulated as a problem in BRST cohomology; see eqs. (4.72), (4.74) and (4.86–89): Let \(\{\vartheta^i\}_{i=1}^n\) be a basis of 1-forms dual to the “vector fields” \(\{L_i\}_{i=1}^n\). Let
\[
\begin{align*}
c^j &:= \vartheta^j \wedge, \quad b_j := L_j \rightarrow .
\end{align*}
\] (5.116)
Then
\[
\begin{align*}
\{c^i, c^j\} = \{b_i, b_j\} &= 0, \quad \{c^i, b_j\} = \delta^i_j,
\end{align*}
\] (5.117)
see (4.71), (4.72). The BRST operator was given by
\[
Q_{\text{BRST}} \equiv Q_{d\pi} = c^j L_j - \frac{1}{2} f^k_{ij} c^i c^j b_k,
\] (5.118)
the \(Z\)-grading operator \(T\) (“ghost number operator”) by
\[
T = c^j b_j .
\] (5.119)
Then
\[
Q_{\text{BRST}}^2 = 0 ,
\] (5.120)
and the \(G\)-invariant states in \(\mathcal{H}\) span the 0th cohomology space, \(H^0_{d\pi}\), of \(Q_{\text{BRST}}\).

This theory has a natural extension to a cohomology theory for graded Lie algebras with values in a representation. Let \(\{\vartheta^1, \ldots, \vartheta^n, \delta^1, \ldots, \delta^m\}\) be a basis dual to \(\{L_1, \ldots, L_n, D_1, \ldots, D_m\}\). As above, we set \(c^j = \vartheta^j \wedge, \ b_j = L_j \rightarrow\), and define
\[
\gamma^j = \delta^j \otimes_s , \quad \beta_j = D_j \rightarrow .
\] (5.121)
Then
\[ [\gamma^i, \gamma^j] = [\beta^i, \beta^j] = 0, \quad [\beta^j, \gamma^i] = \delta^i_j, \] (5.122)
for all \( i \) and \( j \), and the \( \gamma \)'s and \( \beta \)'s commute with the \( c \)'s and the \( b \)'s.

A BRST operator is defined by
\[ Q_{\text{BRST}} = c^j L_j - \frac{1}{2} f_{jk}^{\ i} c^i c^j b_k - g_{ij}^k c^i \gamma^j \beta_k + \gamma^j D_j - \frac{i}{2} h^{kj}_{ij} \gamma^i \gamma^j b_k \] (5.123)
satisfying
\[ Q_{\text{BRST}}^2 = 0, \] (5.124)
and the \( \mathbb{Z} \)-grading operator \( T \) ("ghost number operator") is now given by
\[ T = c^j b_j + \gamma^j \beta_j. \] (5.125)

Of course, in all these formulas, the summation convention is assumed.

Let \( F \cong S((G_{\text{odd}})^*) \otimes \Lambda((G_{\text{even}})^*) \) denote the Fock space on which the canonical anti-commutation relations (5.117) and the canonical commutation relations (5.122) are represented. This representation on \( F \) is uniquely characterized by the property that \( T \) defines a positive, self-adjoint operator on \( F \) with 0 as a simple eigenvalue. The corresponding eigenvector, \( \varphi_0 \), is called the "vacuum". We define
\[ \tilde{\mathcal{H}} := \mathcal{H} \otimes F. \] (5.126)

The space \( \tilde{\mathcal{H}} \) is a \( \mathbb{Z} \)-graded complex for \( Q_{\text{BRST}} \). The 0th cohomology space of \( Q_{\text{BRST}} \) consists precisely of the \( G_{D,\gamma} \)-invariant vectors in \( \mathcal{H} \) (tensored by \( \varphi_0 \)).

Of course, as long as \( \dim(G_{D,\gamma}) < \infty \), the formalism developed here looks somewhat pompous. But its usefulness becomes manifest when \( \dim(G_{D,\gamma}) = \infty \), as is the case e.g. in superstring theory [29].

It is quite clear how this theory can be generalized to non-commutative geometries described by \( N = n^# \) or \( N = (n,n)^# \) supersymmetric spectral data for arbitrary \( n \), where \( n^# \) denotes \( n \) or \( \pi \), etc. We briefly digress on the example of \( N = (1,1) \) (or \( N = (1,1) \)) spectral data,
\[ (\mathcal{A}, \mathcal{H}, d, \tilde{\gamma}, \ast) \simeq (\mathcal{A}, \mathcal{H}, D, \gamma, \bar{D}, \bar{\gamma}) \] ,
with \( D = d + d^* \), \( \gamma = \ast \), \( \bar{D} = i(d - d^*) \). Let \( \mathcal{G} \) be a Lie algebra of infinitesimal reparametrizations of \( (\mathcal{A}, \mathcal{H}, d, \tilde{\gamma}, \ast) \) in the sense of Definition A, above, but assuming in addition that \( L_1, \ldots, L_n \) commute with \( \gamma \) and with \( \bar{\gamma} \). In this situation, a new phenomenon can appear: It may happen that \( d\pi(\mathcal{G}) \) commutes with \( d \). More precisely, let us assume that, for every \( L \in d\pi(\mathcal{G}) \), there exists an element \( X \in \Omega^1_{d\pi}(\mathcal{A}) \), a "vector field", such that
\[ L = \{d, X\}, \] (5.127)
i.e., \( L \) is the Lie derivative associated with \( X \). Then
\[ [d, L] = 0 \quad \text{for all } L \in d\pi(\mathcal{G}). \] (5.128)
If (5.127) holds for all \( L \in d\pi(G) \) we say that \( G \) is a Lie algebra of “infinitesimal diffeomorphisms”, or “vector fields”.

We also assume that the representation \( d\pi \) of \( G \) can be integrated to a unitary representation \( \pi \) of a group \( G \), with \( \text{Lie} \ G = \mathcal{G} \), such that

\[
\pi(g) a \pi(g^{-1}) \in \mathcal{A},
\]

(5.129)

for all \( g \in G \) and all \( a \in \mathcal{A} \) (i.e., \( \pi(G) \) acts by \(*\)-automorphisms on \( \mathcal{A} \)). It then becomes meaningful to study the \( G \)-equivariant \textit{cohomology} of the non-commutative space described by the spectral data \((\mathcal{A}, \mathcal{H}, d, \gamma, \ast)\). In the Cartan model, the \( G \)-equivariant cohomology of \((\mathcal{A}, \mathcal{H}, d, \gamma, \ast)\) can be calculated as follows: Let \( \mathcal{G}^* \) be the dual of \( \mathcal{G} \), with a basis \( \{\gamma^1, \ldots, \gamma^n\} \) dual to the basis \( \{T_1, \ldots, T_n\} \) of \( \mathcal{G} \). Let \( S(\mathcal{G}^*) \) denote the symmetric tensor algebra over \( \mathcal{G}^* \). The algebra \( S(\mathcal{G}^*) \otimes \Omega^*_d(\mathcal{A}) \) carries a natural representation of \( \mathcal{G} \) (and of \( G \)) generated by \textit{Lie derivatives}:

\[
L_T \gamma^k := - f^k_{ij} \gamma^j, \quad \gamma^j \in \mathcal{G}^*,
\]

where \( f^k_{ij} \) are the structure constants of \( \mathcal{G} \); and

\[
L_T \alpha = [L_i, \alpha],
\]

for all \( \alpha \in \Omega^*_d(\mathcal{A}), i = 1, \ldots, n \). One then defines

\[
\mathcal{L}_i := L_{T_i} \otimes 1 + 1 \otimes L_{T_i}.
\]

By \( (S(\mathcal{G}^*) \otimes \Omega^*_d(\mathcal{A}))_{\text{inv}} \) we denote the \( G \)-invariant subalgebra of \( S(\mathcal{G}^*) \otimes \Omega^*_d(\mathcal{A}) \), consisting of all elements which commute with the \( \mathcal{L}_i \).

The algebra \( S(\mathcal{G}^*) \otimes \Omega^*_d(\mathcal{A}) \) is represented on the Hilbert space \( \mathcal{F} \otimes \mathcal{H} \), where \( \mathcal{F} \cong S(\mathcal{G}^*) \) is the symmetric Fock space over \( \mathcal{G}^* \). On this Hilbert space, we introduce the Cartan differential, \( d_C \), by

\[
d_C := 1 \otimes d - \gamma^i \otimes X_i;
\]

(5.130)

note that the degree of \( \gamma^i \) is +2, the one of \( X_i \) is −1. For \( \alpha \in S(\mathcal{G}^*) \otimes \Omega^*_d(\mathcal{A}) \), we define

\[
\delta_C \alpha := [d_C, \alpha]_g.
\]

(5.131)

The \( G \)-equivariant cohomology of \((\mathcal{A}, \mathcal{H}, d, \gamma, \ast)\) is defined by

\[
H^*_{d,G}(\mathcal{A}) := H^*_{dC} \left( (S(\mathcal{G}^*) \otimes \Omega^*_d(\mathcal{A}))_{\text{inv}} \right).
\]

(5.132)

If supersymmetry is “unbroken”, in the sense that there exists a unique vector \( \varphi_0 \in \mathcal{F} \otimes \mathcal{H} \) of degree 0 which is cyclic and separating for the algebra \( S(\mathcal{G}^*) \otimes \Omega^*_d(\mathcal{A}) \) and satisfies \( d \varphi_0 = 0 \), then \( H^*_{d,G}(\mathcal{A}) \) is given by \( H^*_{d,\text{Rham}}(C(\mathcal{E}G) \otimes \mathcal{A}) \) – see the lectures by J.-L. Loday for details.

There is also a BRST model of \( G \)-equivariant cohomology, where \( S(\mathcal{G}^*) \) is replaced by the Weil algebra \( W(\mathcal{G}) = S(\mathcal{G}^*) \otimes \Lambda \mathcal{G}^* \) and \( d_C \) by a differential involving \textit{fermion} creation operators \( c^i \) (besides the \textit{bosonic} \( \gamma^i \)); see e.g. [62].

This theory can be extended to \( N=(2,2) \) spectral data \((\mathcal{A}, \mathcal{H}, \partial, T, \partial, \overline{T}, \ast)\). One then studies e.g. anti-holomorphic infinitesimal reparametrizations generated by operators \( L = \{\partial, X\} \) – hence \( [L, \partial] = 0 \) – which also satisfy \( [L, \overline{T}] = 0 \). The objective is to determine
the Dolbeault cohomology of $\bar{\partial}$ equivariant with respect to a symmetry group generated by operators $L$ with the properties just described.

Cohomological topological quantum theory includes the study of quantum theories consisting of the data $(A, H, Q, G)$, where $Q$ is an operator satisfying $Q^2 = 0$ such that $(H, Q)$ is a $\mathbb{Z}_2$– or $\mathbb{Z}$–graded complex (i.e., there is a grading operator $\gamma$ or $T$, as above), e.g. $Q = Q_{BRST}$, $\bar{\partial}, \partial, \ldots$, and $G$ is a Lie algebra of infinitesimal reparametrizations represented on $H$, with the property that, for each $L \in d\pi(G)$, there exists an odd operator $X$ on $H$ such that $L = \{Q, X\}$. The object of study is the algebra $H_{Q,G}^\bullet(A)$ of cohomology classes and the dual closed currents as considered in subsection 7) of Sect. 5.2. Interesting results emerge from the study of the relations between $H_{\partial,\bar{\partial},G}^\bullet(A)$ and $H_{\bar{\partial},\partial,G}^\bullet(B)$. Among the most interesting ones are those found in examples where $G$ is the Witt algebra of infinitesimal reparametrizations (vector fields) of $S^1$ ($d\pi$ is a projective representation of $G$), where one is led to studying $\text{Diff}(S^1)$–equivariant cohomology in the framework of two-dimensional quantum field theory [63].

The last topic briefly addressed in this section is “target space supersymmetry”. This is a generalization of the notion of spectral data admitting Lie algebras of infinitesimal reparametrizations, as discussed at the beginning of this section. We start, for example, from $N = n$ supersymmetric spectral data $(A, H, D_1, \ldots, D_n, \gamma)$. Let $G$ be a Lie algebra, and $L$ a projective representation of $G$ on $H$ with $\text{ad}L \in \text{Der}(A)$:

$$[L_X, L_Y] = L_{[X,Y]} + C_{X,Y}, \quad (5.133)$$

where $C_{X,Y}$ is an operator $G$–cocycle commuting with $L(G)$ and with $D_1, \ldots, D_n$. We assume that $D_1, \ldots, D_n$ span a module for $G$ in the sense that there are operators

$$\{\lambda^i_j(X) \mid i, j = 1, \ldots, n, \ X \in G\} \quad (5.134)$$

on $H$ which commute with $\{L_X : X \in G\}$ and with $D_1, \ldots, D_n$ and satisfy

$$\sum_j \left( \lambda^i_j(Y) \lambda^k_j(X) - \lambda^i_j(X) \lambda^k_j(Y) \right) = \lambda^k_i([X,Y]) \quad (5.135)$$

as well as

$$[L_X, D_i] = \lambda^i_j(X) D_j, \quad (5.136)$$

for all $X, Y \in G$ and $i = 1, \ldots, n$. Finally,

$$\{D_i, D_j\} = \Delta_{ij}, \quad (5.137)$$

where $\Delta = (\Delta_{ij})_{i,j=1,\ldots,n}$ is a symmetric $n \times n$ matrix of operators on $H$ with the property that

$$[D_k, \Delta_{ij}] = 0, \quad (5.138)$$

for all $i, j$ and $k$. The relations (5.137) and (5.138) generalize analogous relations in (5.22), Sect. 5.1.

Since $D_1, \ldots, D_n$ commute with the $\lambda^i_j(X)$, so does $\Delta_{kl}$, for all $k, l$. Equations (5.136) and (5.137) are compatible with each other iff

$$[L_X, \Delta_{ij}] = \lambda^i_j(X) \Delta_{ij} + \lambda^k_j(X) \Delta_{ik}, \quad 76$$
but this is a consequence of the graded Jacobi identity (5.86):

\[
[L_X, \triangle_{ij}] = [L_X, \{D_i, D_j\}] = \{(L_X, D_i), D_j\} + \{[L_X, D_j], D_i\} = \lambda_i^j(X) \{D_i, D_j\} + \lambda_j^k(X) \{D_k, D_i\} = \lambda_i^j(X) \triangle_{ij} + \lambda_j^k(X) \triangle_{ik}.
\]

Thus \{\triangle_{ij} : i, j = 1, \ldots, n\} span a module for \(\mathcal{G}\) which carries the tensor product of the representation \(\lambda\) and the contragredient representation \(\lambda^\vee\) of \(\mathcal{G}\).

Since the operators \(D_1, \ldots, D_n\) are self-adjoint, and assuming that \(L\) is unitary in the sense that \(L_X^* = -L_X\), one should require that \(\lambda\) is a real representation of \(\mathcal{G}\) in the sense that

\[
\lambda_i^j(X)^* = \lambda_i^j,
\]

for all \(i, j\) and all \(X\) in \(\mathcal{G}\). Furthermore, since \(D_i^* = D_i\) and \(\{D_i, D_j\} = \{D_j, D_i\}\), we have \(\triangle_{ij}^* = \triangle_{ij} = \triangle_{ji}\) for all \(i, j\). This together with (5.138) shows that \{\triangle_{ij}\}_{i,j=1,\ldots,n}\) is a family of commuting, self-adjoint operators on \(\mathcal{H}\) which commute with \(D_k, k = 1, \ldots, n\).

We rewrite the matrix \(\triangle = (\triangle_{ij})\) of commuting operators in the form

\[
\triangle = \frac{1}{n} \left( \sum_{j=1}^n \triangle_{jj} \right) \cdot 1 + \triangle^0,
\]

where \(\sum_{j=1}^n \triangle^0_{jj} = 0\). We then define the subspace \(\mathcal{H}_0\) of \(\mathcal{H}\) to be the eigenspace of the commuting operators \(\triangle^0\) corresponding to the eigenvalue 0. Since the operators \(\triangle^0_{ij}\) commute with \(D_k, k = 1, \ldots, n\), and with \(\triangle_{jj}\), the space \(\mathcal{H}_0\) is invariant under \(D_k\) and \(\triangle_{jj}\). (Note, however, that we do not claim that \(\mathcal{H}_0\) is invariant under \(L_X\), for all \(X \in \mathcal{G}\).)

If we restrict the operators \(D_k\) to \(\mathcal{H}_0\) they satisfy the standard \(N = n\) supersymmetry algebra, i.e., relations (5.22), with \(H = \frac{1}{2n} \left( \sum_{j=1}^n \triangle_{jj} \right)\).

The theory sketched here corresponds to what physicists may call target space supersymmetry. It has a straightforward extension to spectral data of the form \((\mathcal{A}, \mathcal{H}, \{D_i\}_{i=1}^n, \{\bar{D}_i\}_{i=1}^n, \gamma, \mathcal{G})\), which is important when one studies non-real representations of \(\mathcal{G}\) on the complex vector space spanned by \(D_1, \ldots, D_n, \bar{D}_1, \ldots, \bar{D}_n\).

Target space supersymmetries appear in the study of superstring theory in the Green-Schwarz formalism [29] (which exploits the fact that \(SO(8)_v \cong SO(8)_s \cong SO(8)_j\)), and in \(M\) (membrane) theory [61]. The Lie algebra \(\mathcal{G}\) is then supposed to describe some infinitesimal symmetries of “target space”.

For example, \(\mathcal{G}\) could be the Lie algebra of the group of Lorentz transformations of Minkowski space-time. In this case, only the subalgebra of \(\mathcal{G}\) corresponding to infinitesimal rotations of space leaves the subspace \(\mathcal{H}_0\), introduced above, invariant. The operators \(D_k\) are then interpreted as target space supersymmetry generators.

Our exposition of the general formalism of non-commutative differential geometry in Section 5 has clearly suffered from a lack of concrete examples and applications to physics. These form the subject of the last two sections.
6 The non-commutative torus

In order to show how the formalism of non-commutative geometry works in an explicit example, we discuss one of the simplest non-commutative spaces: the two-dimensional non-commutative torus $T^2_\alpha$ (see [51]).

6.1 Spin geometry ($N = 1$)

To begin with, we describe the $N = 1$ data associated to the classical 2-torus $T^2_0$. By Fourier transformation, the algebra of smooth functions over $T^2_0$ is isomorphic to the Schwarz space $A_0 := S(\mathbb{Z}^2)$ over $\mathbb{Z}^2$, endowed with the (commutative) convolution product:

$$ (a \cdot b)(p) = \sum_{q \in \mathbb{Z}^2} a(q) b(p - q) \quad (6.1) $$

where $a, b \in A_0$ and $p \in \mathbb{Z}^2$. Complex conjugation of functions translates into a $^*$-operation:

$$ a^*(p) = \overline{a(-p)} , \quad a \in A_0 . \quad (6.2) $$

If we choose a spin structure over $T^2_0$ in such a way that the spinors are periodic along the elements of a homology basis, then the associated spinor bundle is a trivial rank 2 vector bundle. With this choice, the space of square integrable spinors is given by the direct sum

$$ H_e \equiv H = l^2(\mathbb{Z}^2) \oplus l^2(\mathbb{Z}^2) \quad (6.3) $$

where $l^2(\mathbb{Z}^2)$ denotes the space of square summable functions over $\mathbb{Z}^2$. The algebra $A_0$ acts diagonally on $H$ by the convolution product. We choose a flat metric $(g_{\mu\nu})$ on $T^2_0$ and we introduce the corresponding 2-dimensional gamma matrices

$$ \{ \gamma^\mu, \gamma^\nu \} = -2 g^{\mu\nu} , \quad \gamma^{\mu*} = -\gamma^\mu . \quad (6.4) $$

Then, the Dirac operator $D$ on $H$ is given by

$$ (D \xi)(p) = i p_\mu \gamma^\mu \xi(p) , \quad \xi \in H . \quad (6.5) $$

Finally, the $\mathbb{Z}_2$-grading $\sigma$ on $H$ can be written as

$$ \sigma = \frac{i}{2} \sqrt{g} \varepsilon_{\mu\nu} \gamma^\mu \gamma^\nu \quad (6.6) $$

where $\varepsilon_{\mu\nu}$ denotes the Levi-Civita tensor. The data $(A_0, H, D, \sigma)$ are the canonical $N = 1$ data associated to the compact spin manifold $T^2_0$, and it is clear that they satisfy all the properties of Definition A in Sect. 5.1.

The non-commutative torus is obtained by deforming the product of the algebra $A_0$. For each $\alpha \in \mathbb{R}$, we define the algebra $A_\alpha := S(\mathbb{Z}^2)$ with the product

$$ (a \bullet_\alpha b)(p) = \sum_{q \in \mathbb{Z}^2} a(q) b(p - q) e^{i \pi \alpha \omega(p,q)} \quad (6.7) $$

where $\omega$ is the integer-valued anti-symmetric bilinear form on $\mathbb{Z}^2 \times \mathbb{Z}^2$

$$ \omega(p,q) = p_1 q_2 - p_2 q_1 , \quad p, q \in \mathbb{Z}^2 . \quad (6.8) $$
The *-operation is defined as before. Alternatively, we could introduce the algebra \( \mathcal{A}_\alpha \) as the unital *-algebra generated by the elements \( U \) and \( V \) subject to the relations
\[
UU^* = U^*U = VV^* = V^*V = 1, \quad UV = e^{-2\pi i \alpha} VU.
\]

Having chosen an appropriate closure, the equivalence of the two descriptions is easily seen if one makes the following identifications:
\[
U(p) = \delta_{p,1} \delta_{p,0}, \quad V(p) = \delta_{p,1} \delta_{p,0}.
\]

If \( \alpha \) is a rational number, \( \alpha = \frac{M}{N} \), where \( M \) and \( N \) are co-prime integers, then the centre \( Z(\mathcal{A}_\alpha) \), of \( \mathcal{A}_\alpha \) is infinite-dimensional:
\[
Z(\mathcal{A}_\alpha) = \operatorname{span}\{ U^{mN}V^{nN} | m, n \in \mathbb{Z} \}.
\]

Let \( I_\alpha \) denote the ideal of \( \mathcal{A}_\alpha \) generated by \( Z(\mathcal{A}_\alpha) - 1 \). Then it is easy to see that the quotient \( \mathcal{A}_\alpha / I_\alpha \) is isomorphic, as a unital *-algebra, to the full matrix algebra \( M_N(\mathbb{C}) \).

If \( \alpha \) is irrational, the centre of \( \mathcal{A}_\alpha \) is trivial and \( \mathcal{A}_\alpha \) is of type II\(_1\), the trace being given by the evaluation at \( p = 0 \). Unless stated differently, we shall only study the case of irrational \( \alpha \) here, but the finite-dimensional non-commutative torus will be used in Sect. 7.3 below.

We define the non-commutative 2-torus \( T^2_\alpha \) by its \( N = 1 \) data \( (\mathcal{A}_\alpha, \mathcal{H}, D, \sigma) \) where \( \mathcal{H}, D \) and \( \sigma \) are as in eqs. (6.3), (6.5) and (6.6), and \( \mathcal{A}_\alpha \) acts diagonally on \( \mathcal{H} \) by the deformed product, eq. (6.7). When \( \alpha = \frac{M}{N} \) is rational, one may work with the data \( (\mathcal{A}_\alpha / I_\alpha, \mathbb{C}^N \oplus \mathbb{C}^N, D_\alpha, \sigma) \), where the Dirac operator \( D_\alpha \) is given by
\[
D_\alpha = i \gamma^\mu \sin \left( \frac{\pi}{N} p^\mu \right).
\]

### 1) Differential forms

Recall that there is a representation \( \pi \) of the algebra of universal forms \( \Omega^*(\mathcal{A}_\alpha) \) on \( \mathcal{H} \) (see Sect. 5.1, subsection 2)). The images of the homogeneous subspaces of \( \Omega^*(\mathcal{A}_\alpha) \) under \( \pi \) are given by
\[
\pi(\Omega^0(\mathcal{A}_\alpha)) = \mathcal{A}_\alpha \quad \text{(by definition)}
\]
\[
\pi(\Omega^{2k-1}(\mathcal{A}_\alpha)) = \{ a_\mu \gamma^\mu | a_\mu \in \mathcal{A}_\alpha \}
\]
\[
\pi(\Omega^{2k}(\mathcal{A}_\alpha)) = \{ a + b \sigma | a, b \in \mathcal{A}_\alpha \}
\]
for all \( k \in \mathbb{Z}_+ \). In principle, one should then compute the kernels \( J^n \) of \( \pi \) (see eq. (5.2)), but these are generally huge and difficult to describe explicitly. To determine the space of \( n \)-forms, it is simpler to use the isomorphism (see eq. (5.3))
\[
\Omega^n_D(\mathcal{A}_\alpha) = \Omega^n(\mathcal{A}_\alpha) / (J^n + \delta J^{n-1}) \cong \pi(\Omega^n(\mathcal{A}_\alpha)) / \pi(\delta J^{n-1}).
\]

The spaces \( \pi(\delta J^{n-1}) \) are easy to compute, and the result is
\[
\pi(\delta J^1) = \mathcal{A}_\alpha
\]
\[
\pi(\delta J^{2k}) = \pi(\Omega^{2k+1}(\mathcal{A}_\alpha))
\]
\[
\pi(\delta J^{2k+1}) = \pi(\Omega^{2k+2}(\mathcal{A}_\alpha))
\]
for all \( k \geq 1 \). The spaces of \( n \)-forms are thus given (up to isomorphism) by

\[
\begin{align*}
\Omega^0_D(A_\alpha) &= A_\alpha \\
\Omega^1_D(A_\alpha) &\cong \{ a_\mu \gamma^\mu \mid a_\mu \in A_\alpha \} \\
\Omega^2_D(A_\alpha) &\cong \{ a \sigma \mid a \in A_\alpha \} \\
\Omega^n_D(A_\alpha) &= 0 \text{ for } n \geq 3
\end{align*}
\] (6.20)

where we have chosen special representatives on the r.s. Notice that \( \Omega^1_D(A_\alpha) \) and \( \Omega^2_D(A_\alpha) \) are free left \( A_\alpha \)-modules of rank 2 and 1, respectively. This reflects the fact that the bundles of 1- and 2-forms over the 2–torus are trivial and of rank 2 and 1, resp.

2) **Integration and Hermitian structure over** \( \Omega^1_D(A_\alpha) \)

It follows from eqs. (6.13–15) that there is an isomorphism \( \pi(\Omega^\ast(D(A_\alpha))) \cong A_\alpha \otimes M_2(\mathbb{C}) \). Applying the general definition of the integral — see Sect. 5.1, 3) — to the non-commutative torus, one finds

\[
\int \omega = \text{Tr}_{C^2}(\omega(0))
\] (6.24)

for an arbitrary element \( \omega \in \pi(\Omega^\ast(D(A_\alpha))) \). The cyclicity property, Assumption A in Sect. 5.1, 3), follows directly from the definition of the product in \( A_\alpha \) and the cyclicity of the trace on \( M_2(\mathbb{C}) \). The kernels \( K^n \) of the canonical sesqui-linear form on \( \pi(\Omega^\ast(D(A_\alpha))) \) — see eq. (5.5) — coincide with the kernels \( J^n \) of \( \pi \), and we get for all \( n \in \mathbb{Z} \):

\[
\tilde{\Omega}^n(A_\alpha) = \Omega^n(A_\alpha), \quad \tilde{\Omega}^n_D(A_\alpha) = \Omega^n_D(A_\alpha).
\] (6.25)

Note that the equality \( K^n = J^n \) holds in all explicit examples of non-commutative \( N = 1 \) spaces studied so far. It is easy to see that the canonical representatives \( \omega^\perp \) on \( H \) of differential forms \( [\omega] \in \Omega^n_D(A_\alpha) \), see eq. (5.10), coincide with the choices already made in eqs. (6.20–23). The canonical Hermitian structure on \( \Omega^1_D(A_\alpha) \) is given by

\[
\langle \omega, \eta \rangle_D = \omega_\mu g^{\mu\nu} \eta^\nu \in A_\alpha
\] (6.26)

for all \( \omega, \eta \in \Omega^1_D(A_\alpha) \). Note that this is a true Hermitian metric, i.e., it takes values in \( A_\alpha \) and not in the weak closure \( A_\alpha'' \). Again, this is true in many in other examples, as well.

3) **Connections on** \( \Omega^1_D(A_\alpha) \)

Since \( \Omega^1_D(A_\alpha) \) is a free left \( A_\alpha \)-module, it admits a basis which we can choose to be \( E^\mu := \gamma^\mu \). A connection \( \nabla \) on \( \Omega^1_D(A_\alpha) \) is uniquely specified by its coefficients \( \Gamma^\lambda_{\mu\nu} \in A_\alpha \),

\[
\nabla E^\mu = -\Gamma^\mu_{\nu\lambda} E^\nu \otimes E^\lambda \in \Omega^1_D(A_\alpha) \otimes_{A_\alpha} \Omega^1_D(A_\alpha),
\] (6.27)

and these coefficients can be chosen arbitrarily. Note that in the classical case (\( \alpha = 0 \)) the basis \( E^\mu \) consists of real 1-forms. Thus, we say that the connection \( \nabla \) is real if its coefficients in the basis \( E^\mu \) are self-adjoint elements of \( A_\alpha \). A simple computation shows that there is a unique real, unitary, torsionless connection \( \nabla^{L.C.} \) on \( \Omega^1_D(A_\alpha) \) given by

\[
\nabla^{L.C.} E^\mu = 0.
\] (6.28)
6.2 Riemannian geometry \((N = (1, 1))\)

In this subsection, we derive a set of \(N = (1, 1)\) spectral data along the lines of Sect. 5.2, subsection 5). Our first task is to find a real structure \(J\) on the \(N = 1\) data \((\mathcal{A}_\alpha, \mathcal{H}, D, \sigma)\). To this end, we introduce the complex conjugation \(\kappa : \mathcal{H} \rightarrow \mathcal{H}\), \((\kappa \xi)(p) := \bar{\xi}(p)\), as well as the charge conjugation matrix \(C : \mathcal{H} \rightarrow \mathcal{H}\) as the unique (up to a sign) constant matrix such that

\[
C \gamma^\mu = -\bar{\gamma}^\mu C \\
C = C^\ast = C^{-1}.
\]  

Then the most natural real structure which satisfies \([JaJ^\ast, b] = 0\) for all \(a, b \in \mathcal{A}_\alpha\) as well as \([J, D] = 0\) is simply given by

\[
J := C\kappa.
\]  

The right actions of \(\mathcal{A}_\alpha\) and \(\Omega^1_D(\mathcal{A}_\alpha)\) on \(\mathcal{H}\) (see 5.2.5) are given as follows

\[
\xi \ast a = J a^\ast J^\ast \xi = \xi \ast a^\gamma \\
\xi \ast \omega = J \omega^\ast J^\ast \xi = \gamma^\mu \xi \ast a^\gamma
\]

where \(\xi, \omega \in \mathcal{H}\), \(a \in \mathcal{A}_\alpha\), \(\omega \in \Omega^1_D(\mathcal{A}_\alpha)\), \(\xi \ast a\) denotes the diagonal right action of \(a\) on \(\xi\)

by the deformed product, and

\[
a^\gamma(p) := a(-p).
\]

Notice that \((a \ast b)^\gamma = a^\gamma \ast b^\gamma\). We denote by \(\mathcal{H}^\circ\) the dense subspace \(S(\mathbb{Z}^2) \oplus S(\mathbb{Z}^2) \subset \mathcal{H}\) of smooth spinors. The space \(\mathcal{H}^\circ\) is a two-dimensional free left \(\mathcal{A}_\alpha\)--module with canonical basis \(\{e_1, e_2\}\). Then any connection \(\nabla\) on \(\mathcal{H}^\circ\) is uniquely determined by its coefficients \(\omega_j^i \in \Omega^1_D(\mathcal{A}_\alpha)\):

\[
\nabla e_i = \omega_j^i \otimes e_j = \omega_j^i \gamma^\mu \otimes e_j \in \Omega^1_D(\mathcal{A}_\alpha) \otimes \mathcal{A}_\alpha \mathcal{H}.
\]  

The “associated right connection” \(\nabla^\ast\) is then given by

\[
\nabla^\ast e_i = e_j \otimes \tilde{\omega}_j^i \in \mathcal{H}^\circ \otimes \mathcal{A}_\alpha \Omega^1_D(\mathcal{A}_\alpha)
\]

where

\[
\tilde{\omega}_j^i = -C^i_k(\omega^k_\ast \ast C^l_j = C^i_k(\omega^k_\ast) \ast C^l_j \gamma^\mu.
\]  

An arbitrary element in \(\mathcal{H}^\circ \otimes \mathcal{A}_\alpha \mathcal{H}\) can be written as \(e_i \otimes a^{ij} e_j\) where \(a^{ij} \in \mathcal{A}_\alpha\). The “Dirac operators” \(\mathcal{D}\) and \(\bar{\mathcal{D}}\) on \(\mathcal{H}^\circ \otimes \mathcal{A}_\alpha \mathcal{H}\) associated to the connection \(\nabla\) are given by (see eq. (5.42))

\[
\mathcal{D}(e_i \otimes a^{ij} e_j) = e_i \otimes (\delta a^{ij} + \tilde{\omega}_k^i a^{kj} + a^{ik} \omega_k^j) \ast e_j
\]

\[
\bar{\mathcal{D}}(e_i \otimes a^{ij} e_j) = e_i \ast (\delta a^{ij} + \tilde{\omega}_k^i a^{kj} + a^{ik} \omega_k^j) \ast \sigma e_j.
\]

In order to be able to define a scalar product on \(\mathcal{H}^\circ \otimes \mathcal{A}_\alpha \mathcal{H}\), we need a Hermitian structure on the right module \(\mathcal{H}\), denoted by \(\langle \cdot, \cdot \rangle\), with values in \(\mathcal{A}_\alpha\). It is defined by

\[
\int \langle \xi, \zeta \rangle a = \langle \xi, \zeta a \rangle \quad \forall \xi, \zeta \in \mathcal{H}^\circ, \forall a \in \mathcal{A}_\alpha.
\]  

81
This Hermitian structure can be written explicitly as

$$\langle \xi, \zeta \rangle = \sum_{i=1}^{1} \xi_i \cdot \alpha \zeta_i \vee,$$  

(6.40)

and it satisfies

$$\langle a \xi, b \zeta \rangle = a^* \langle \xi, \zeta \rangle b$$

(6.41)

for all $\xi, \zeta \in \mathcal{H}$ and $a, b \in \mathcal{A}_\alpha$. Then we define the scalar product on $\mathcal{H} \otimes \mathcal{A}_\alpha \mathcal{H}$ as (see [5])

$$\langle \xi_1 \otimes \xi_2, \zeta_1 \otimes \zeta_2 \rangle = \langle \xi_2, \langle \xi_1, \zeta_1 \rangle \zeta_2 \rangle.$$  

(6.42)

This expression can be written in a more suggestive way if one introduces a Hermitian structure, denoted $\langle \cdot, \cdot \rangle_L$, on the left module $\mathcal{H}$:

$$\langle \xi, \zeta \rangle_L := \langle J \xi, J \zeta \rangle.$$  

This Hermitian structure satisfies

$$\langle a \xi, b \zeta \rangle_L = a \langle \xi, \zeta \rangle_L b^*$$

for all $a, b \in \mathcal{A}_\alpha$ and $\xi, \zeta \in \mathcal{H}$, and the scalar product on $\mathcal{H} \otimes \mathcal{A}_\alpha \mathcal{H}$ can be written as follows

$$\langle \xi_1 \otimes \xi_2, \zeta_1 \otimes \zeta_2 \rangle = \int \langle \xi_1, \zeta_1 \rangle \langle \zeta_2, \xi_2 \rangle_L.$$  

This expression can be written in a more suggestive way if one introduces a Hermitian structure, denoted $\langle \cdot, \cdot \rangle_L$, on the left module $\mathcal{H}$:

$$\langle \xi, \zeta \rangle_L := \langle J \xi, J \zeta \rangle.$$  

A tedious computation shows that the relations

$$D^* = D, \quad \bar{D}^* = \bar{D}, \quad \{D, \bar{D}\} = 0, \quad D^2 = \bar{D}^2$$

(6.43)

are equivalent to

$$\nabla e_i = 0 \quad \forall i.$$  

(6.44)

In particular, we see that the original $N = 1$ data uniquely determine the operators $D$ and $\bar{D}$ satisfying the $N = (1, 1)$ algebra, eq. (6.43).

One can prove that the $\mathbb{Z}_2$-grading operators $\gamma$ and $\bar{\gamma}$ (see Sect. 5.2, subsection 1)) are also unique (up to a sign):

$$\gamma = 1 \otimes \sigma, \quad \bar{\gamma} = \sigma \otimes 1.$$  

(6.45)

In summary, we see that we get a natural set of $N = (1, 1)$ data $(\mathcal{A}_\alpha, \mathcal{H} \otimes \mathcal{A}_\alpha \mathcal{H}, D, \gamma, \bar{D}, \bar{\gamma})$ induced by the original $N = 1$ data. Furthermore, there is a unique operator $T$,

$$T = \frac{1}{2i} g_{\mu\nu} \gamma^\mu \otimes \gamma^\nu \sigma$$  

(6.46)

that makes $(\mathcal{A}_\alpha, \mathcal{H} \otimes \mathcal{A}_\alpha \mathcal{H}, D, \gamma, \bar{D}, \bar{\gamma}, T)$ into a set of $N = (1, 1)$ data, as defined at the end of Sect. 5.2, subsection 1).
6.3 Kähler geometry \((N = (2,2))\)

In this subsection, we extend the \(N = (1,1)\) spectral data to \(N = (2,2)\) data. The simplest way to construct this extension is to determine all anti-selfadjoint operators, collectively denoted by \(I\), that commute with \(A_\alpha, \gamma, \bar{\gamma}\) and \(T\) (see subsection 5.2.9). Then one defines the additional differentials as in eq. (5.91). The most general operator \(I\) on \(\mathcal{H} \otimes A_\alpha \mathcal{H}\) that commutes with all elements of \(A_\alpha\) is of the form

\[
I = \sum_{\mu, \nu = 0}^{3} \gamma^\mu \otimes \gamma^\nu I^R_{\mu\nu}
\]

where \(I^R_{\mu\nu}\) are elements of \(A_\alpha\) acting on \(\mathcal{H} \otimes A_\alpha \mathcal{H}\) from the right, and where we have set

\[
\gamma^0 = 1, \quad \gamma^3 = \sigma.
\]

The vanishing of the commutators of \(I\) with \(\gamma\) and \(\bar{\gamma}\) implies that \(I^R_{\mu\nu} = 0\) unless \(\mu, \nu \in \{0,3\}\). The equation \([I, T] = 0\) requires \(I^R_{03} = I^R_{30}\) and leaves the coefficients \(I^R_{00}\) and \(I^R_{33}\) undetermined. Since the operators \(I\) appear only through commutators, their trace part is irrelevant and we can set \(I^R_{00} = 0\). All constraints together give

\[
I = (\sigma \otimes 1 + 1 \otimes \sigma) I^R_{03} + (\sigma \otimes \sigma) I^R_{33}
\]

where \(I^R_{03}\) and \(I^R_{33}\) are anti-selfadjoint elements of \(A_\alpha\). We decompose \(I\) into two parts

\[
I_1 = (\sigma \otimes 1 + 1 \otimes \sigma) I^R_{03}
\]

\[
I_2 = (\sigma \otimes \sigma) I^R_{33}
\]

and we introduce the new differentials according to eq. (5.91)

\[
d_1 = d = D - i \bar{D}
\]

\[
d_2 = [I_1, d]
\]

\[
d_3 = [I_2, d].
\]

The nilpotency of \(d_2\) and \(d_3\) implies that \(I_{03}\) and \(I_{33}\) are multiples of the identity, and we normalize them as follows

\[
I_1 = \frac{i}{2} (\sigma \otimes 1 + 1 \otimes \sigma)
\]

\[
I_2 = i (\sigma \otimes \sigma).
\]

Comparing eqs. (6.56) and (6.45) we see that

\[
I_2 = i \gamma \bar{\gamma}
\]

and it follows, using eqs. (6.52) and (6.54), that

\[
d_3 = [I_2, d] = 2 i d \gamma \bar{\gamma}.
\]

Thus, the differential \(d_3\) is a trivial modification of \(d\), and we discard it. It is then easy to verify that \((A_\alpha, \mathcal{H} \otimes A_\alpha, \mathcal{H}, d_1, d_2, \gamma, \bar{\gamma}, T, I_1)\) form a set of \(N = (2,2)\) spectral data.
Furthermore, they are, as we have shown, uniquely determined by the original $N = (1,1)$ data. Therefore, a Riemannian non-commutative torus (at irrational deformation parameter $\alpha$) admits a unique Kähler structure.

We have only given the definitions of the spectral data in the $N = (1,1)$ and the $N = (2,2)$ setting. As a straightforward application of the general methods described in Section 5, we could compute the associated de Rham resp. Dolbeault complexes, as well as the Euler characteristic, the Hirzebruch signature, or geometrical quantities like curvature. We do not carry out these calculations here.

Instead, we emphasize the following feature: For rational deformation parameter $\alpha = \frac{M}{N}$, the algebra $\mathcal{A}_\alpha$ in itself does not specify the geometry of the underlying non-commutative space. It is only the selection of a specific $K$–cycle $(\mathcal{H},D)$ that allows us to identify this space as a deformed torus. In fact, by picking different pairs $(\mathcal{H},D)$ for $\mathcal{A}_\alpha = M_N(\mathbb{C})$, one can obtain the fuzzy two-sphere, and even the fuzzy three-sphere (see Sec. 7.6).

One might speculate that for irrational $\alpha$ the choice of $K$–cycles is more restricted. It would be very interesting to investigate how the geometries for $\mathcal{A}_\alpha, \alpha \notin \mathbb{Q}$, can be approximated by “towers of matrix geometries”.

7 Applications of non-commutative geometry to quantum theories of gravitation

In this section we sketch some applications of the tools described in Sections 4–6 to a quantum theory of gravitation yet to be discovered. We have argued in Section 3 that a combination of quantum theory and general relativity leads to the prediction that space-time cannot be a classical manifold and that the basic degrees of freedom of a theory of space-time-matter had better be associated with extended objects so that space-time uncertainty relations valid for the location of events are automatically fulfilled. A currently popular idea is that those extended objects are strings. We therefore sketch some features of string theory; for a broad exposition of the subject see [29]. However, the consensus evolves in the direction to say that there are extended dynamical objects, “branes”, of various dimensions and that, perhaps, extended objects more fundamental than strings might be “membranes” ($M$–theory); see Sect. 7.3.

7.1 From point-particles to strings

Let $M$ be a $d$–dimensional, Lorentzian manifold interpreted as classical space-time. We consider a point-particle moving in $M$, as discussed in Section 1. But now we propose to treat it quantum mechanically, following Feynman’s idea of path integrals. The action of a relativistic point-particle is given by (see eq. (1.6))

$$S_P(x,h) := \frac{1}{2} \int_0^1 g_{\mu\nu}(x(\tau)) \dot{x}^\mu(\tau) \dot{x}^\nu(\tau) h(\tau)^{-1/2} \, d\tau$$
\[
\frac{\mu^2 l^{-1}}{2} \int_0^1 h(\tau)^{1/2} \, d\tau ,
\]  
(7.1)

where \((g_{\mu\nu})\) is a Lorentzian metric on \(M\), \(\dot{x}(\tau) := \frac{dx(\tau)}{d\tau}\), \(h(\tau)\) is a metric on the unit interval \([0, 1]\); \(\mu^2\) is a positive constant of dimension mass\(^2\), and \(l\) is a constant with the dimension of length. Feynman proposed to consider a path integral related to

\[
\Delta_F(x, y) := \int_{\substack{x(0) = x \\ x(1) = y}} e^{iS_P(x, h)} \, Dx \, Dh ,
\]  
(7.2)

where, formally, \(Dx = \prod_{\tau \in [0, 1]} \frac{dx(\tau)}{l} \), \(Dh = \prod_{\tau \in [0, 1]} dh(\tau)\). Choosing a gauge such that \(h(\tau) \equiv T^2, 0 < T < \infty\), and performing the \(x\)-integral, one finds that

\[
\Delta_F(x, y) = \text{const} \cdot \int_0^\infty dT \left( e^{iT\left(\Box_g + \mu^2 + i0\right)} \right) (x, y)
\]

\[
= \text{const} \cdot l^{-1} \left( \Box_g + \mu^2 + i0 \right)^{-1} (x, y) .
\]

This is the Feynman propagator for a scalar particle with mass \(\mu\). For \(y^0 > x^0\), \(\Delta_F(x, y)\) is a matrix element of the quantum-mechanical particle propagator from time \(x^0\) to time \(y^0\). Unfortunately, it is not very meaningful to consider a single point-particle. First, the principles of local, relativistic quantum field theory imply that every particle has a twin, the anti-particle (possibly identical with the particle), and, second, when the metric \((g_{\mu\nu})\) on \(M\) is not static (but \(M\) is asymptotically Minkowskian) then particle creation- and annihilation processes are observed (for a physically meaningful definition of particles).

So we really must consider a gas of particle world-lines. The partition function, \(\Xi\), of this gas is obtained by integrating over all configurations of closed world-lines (i.e., loops), each one weighted by \(\int \exp(iS_P(x, h)) \, Dh\).

According to Symanzik [64], a system of interacting, relativistic, scalar point-particles can be described in terms of a gas of world-lines with local soft-core repulsion (“excluded volume interactions”). This approach has ultimately led to various non-interaction theorems for scalar quantum field theories (triviality of \(\lambda \phi^4\) in \(d \geq 4\))[65].

It is clear that the interactions between different point-particles depend on the way their world-lines are embedded in the classical space-time background. In other words, the formulation of a local quantum theory of interacting, relativistic, scalar point-particles requires a model of classical space-time.

It is not difficult to guess how one might generalize the Feynman-Symanzik formulation of the quantum theory of relativistic, scalar point-particles to a quantum theory of relativistic string-like extended objects. Let us first consider the relativistic mechanics of a classical string: It sweeps out a world-sheet \(X : \Sigma \to M\), \(\Sigma \ni \xi \mapsto (X^\mu(\xi)) \in M\), where \(\Sigma\) is a surface equipped with a (Lorentz) metric \(h = (h_{\alpha\beta}(\xi))\). As the equations of motion for \(h\) and \(X\), Deser, Zumino and Polyakov [66] have proposed the Euler-Lagrange equations corresponding to the following action functional (see also [67]):

\[
S_{P, \Sigma}(X, h) := \frac{1}{4\pi \alpha'} \int_{\Sigma} d^2\xi \sqrt{|h(\xi)|} h^{\alpha\beta}(\xi) \partial_\alpha X^\mu(\xi) g_{\mu\nu}(X(\xi)) \partial_\beta X^\nu(\xi) +
\]
\[ + \frac{\Lambda}{4\pi} \int_{\Sigma} d^2 \xi \sqrt{|h(\xi)|}, \quad (7.3) \]

where \(\alpha'\) and \(\Lambda^{-1}\) are constants of dimension length\(^2\). The solutions to the classical equations of motion are extremal surfaces in \(M\). Actually, the action (7.3) should be generalized by including two further terms:

\[ S_{P, \Sigma}(X, h) \rightarrow S_{\text{tot}, \Sigma}(X, h) := S_{P, \Sigma}(X, h) + S_\Sigma'(X, h) + S_\Sigma''(X, h), \]

where

\[ S_\Sigma' := \frac{1}{4\pi \alpha'} \int_{\Sigma} d^2 \xi \epsilon^{\alpha\beta} \partial_{\alpha} X^\mu(\xi) B_{\mu\nu}(X(\xi)) \partial_{\beta} X^\nu(\xi), \quad (7.4) \]

\[ S_\Sigma'' := \frac{1}{4\pi} \int_{\Sigma} d^2 \xi \sqrt{|h(\xi)|} \Phi(X(\xi)) r(\xi), \quad (7.5) \]

Here \(\beta \equiv B_{\mu\nu}(x) dx^\mu \wedge dx^\nu\) is a 2-form on \(M\) and \(\Phi(x)\), the “dilaton”, is a function on \(M\); \(r(\xi)\) is the curvature scalar corresponding to the metric \((h_{\alpha\beta}(\xi))\) on \(\Sigma\). The term \(S_\Sigma'\) is proportional to the integral of \(\beta\) over the image of \(\Sigma\) under the map \(X : \Sigma \rightarrow M\), \(\Sigma \ni \xi \mapsto X(\xi) \in M\), (which is the integral of the pullback \(X^*(\beta)\) over \(\Sigma\)) and \(S_\Sigma''\) is proportional to the integral of \(X^*(\Phi) r\) over \(\Sigma\).

Let us consider a single, relativistic, closed string propagating from some initial to some final configuration (at larger times). Then \(\Sigma\) has the topology of a cylinder (i.e. of a twice punctured sphere). We are actually interested in the quantum-mechanical propagation of a relativistic, closed string. In analogy to Feynman’s quantization of the mechanics of relativistic, scalar point-particles in terms of path integrals (see (7.2)), one is tempted to guess that the propagator is given by

\[ \Delta_F(X_i, X_f) := \int e^{iS_{\text{tot}, \Sigma}(X, h)} D_h X D h, \quad (7.6) \]

where \(X_i\) and \(X_f\) denote configurations of the string contained in (co-dimension 1) spacelike surfaces \(\sigma_i\) and \(\sigma_f\), respectively, embedded in \(M\) in such a way that \(\sigma_i\) is e.g. earlier than \(\sigma_f\) with respect to the causal orientation of \(M\). They provide boundary conditions for the functional integral on the r.s. of (7.6) at \(\partial \Sigma\).

On the r.s. of (7.6), we invoke Fubini’s theorem to represent \(\Delta_F(X_i, X_f)\) as

\[ \Delta_F(X_i, X_f) = \int D_h X \int D_h e^{iS_{\text{tot}, \Sigma}(X, h)} =: \int D_h \ Z_\Sigma(h). \quad (7.7) \]

Formally, the measure \(D_h X\) is the Riemannian volume form on the infinite-dimensional Riemannian manifold of maps \(X\) from \((\Sigma, h)\) to \((M, g)\) and hence depends on \(h\) (and on \(g\) – but \(g\) is presently kept fixed). If \(\psi\) is a diffeomorphism of \(\Sigma\) onto itself then

\[ Z_\Sigma(h) = Z_\Sigma(\psi^* h). \quad (7.8) \]

It would seem that eq. (7.8) holds by construction. However, since the calculation of \(Z_\Sigma(h)\) involves a formal, infinite-dimensional functional integration, one should ask whether “diffeomorphism (or gravitational) anomalies” could invalidate (7.8). It turns out that if the
field \( X(\xi), \xi = (\sigma, \tau) \in \Sigma, \) is a non-chiral field (in the sense that left-moving modes of \( X, \) depending on \( \sigma + \tau, \) match right-moving ones, depending on \( \sigma - \tau \)) then there are no such anomalies. But if \( X \) were chiral (e.g., purely left-moving) then gravitational anomalies appear. They can be described as Lorentz– and mixed Lorentz–Weyl anomalies and are cancelled by the ones of a three-dimensional gravitational Chern-Simons action \([77]\). (This leads to the prediction that \((M, g)\) should be a Lorentzian manifold of dimension \(26 + n \cdot 24, \ n = 0, 1, \ldots\).)

Eq. (7.8) implies that \( Z_\Sigma(h) \) only depends on the orbit \([h]\) of \( h \) under the pullback action of the group of diffeomorphisms of \( \Sigma. \) Thus the integral (7.7) is ill-defined before we fix a gauge. On a cylinder \( \Sigma, \) the orbit \([h]\) of every metric \( h \) contains a conformally flat metric, \( e^{\phi(\xi)} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \) i.e.,

\[
h_{\alpha\beta}(\xi) \sim e^{\phi(\xi)} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\] (7.9)

Thus, orbit space is parametrized by the conformal factors \( e^{\phi(\xi)} \) (\( \phi \) is called "Liouville mode") and we can choose the conformally flat metrics (r.s. of (7.9)) as a cross section in the Riemannian manifold of all metrics on \( \Sigma. \) (This is what the physicists call a gauge choice.) From (7.9) we conclude that one can equip \( \Sigma \) with a causal structure (a field of light cones), independently of the choice of \((h_{\alpha\beta}), \) and this suggests that a local (w.r.t. the causal structure on \( \Sigma \)) “quantum theory” of the metric \((h_{\alpha\beta})\) can be developed (i.e., two-dimensional quantum gravity ought to make sense – recall the discussion towards the end of Section 3). The functional integral formulation of this “quantum theory” is quite well understood: One fixes the gauge specified on the r.s. of (7.9). We denote

\[
(h_{\alpha\beta}) := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\] (7.10)

Using the Faddeev-Popov method \([68,29]\), one finds (see \([66]\)) that

\[
\triangle_F(X_i, X_f) = \text{const} \int \mathcal{D}h \phi e^{-2\text{Re} \Gamma_h(\phi)} Z_\Sigma(e^{\phi} \hat{h}),
\] (7.11)

where

\[
\Gamma_h(\phi) = \frac{1}{96\pi} \int_\Sigma d^2\xi \sqrt{|\hat{h}(\xi)|} \left( \hat{h}^{\alpha\beta}(\xi) \partial_\alpha \phi(\xi) \partial_\beta \phi(\xi) + 4r_h(\xi) \phi(\xi) \right).
\] (7.12)

Of course, for our gauge choice (7.9), \( |\hat{h}(\xi)| = 1 \) and \( r_h(\xi) \equiv 0; \) but it is useful to know the general result (7.12) for \( \Gamma_h \), in order to be able to extend these calculations to more general surfaces \( \Sigma \) (where \( \hat{h} \) ranges over some moduli space of conformally inequivalent, non-flat — and, for Minkowskian signature, singular — metrics over which one will have to integrate; see e.g. \([87]\)).

Note that (for \( \Lambda = 0 \) on the r.s. of (7.3))

\[
S_{\text{tot},\Sigma}(X, e^{\phi} \hat{h}) = S_{\text{tot},\Sigma}(X, \hat{h}),
\] (7.13)

i.e., \( S_{\text{tot},\Sigma} \) is invariant under Weyl rescaling. One might thus expect that \( Z_\Sigma(e^{\phi} \hat{h}) = Z_\Sigma(\hat{h}) \). However, this is never true, because \( \mathcal{D}_{e^{\phi} \hat{h}}X \) does depend on \( \phi; \) one says that two-dimensional quantum field theories always have a Weyl anomaly. Thus, while we have that

\[
Z_\Sigma(\psi^* h) = Z_\Sigma(h),
\] 87
for all metrics $h$ and all diffeomorphisms $\psi$ of $\Sigma$ (see (7.8)), one always finds that

$$Z_{\Sigma}(e^{\phi} h) \neq Z_{\Sigma}(h) .$$

(7.14)

One way of trying to give meaning to the $\phi$–integral on the r.s. of eq. (7.11) is to demand that the $\phi$–dependence of the integrand be trivial. Then

$$Z_{\Sigma}(e^{\phi} h) = e^{i c \Gamma_{\hat{h}}(\phi)} Z_{\Sigma}(\hat{h}) ,$$

(7.15)

with

$$c = 26 ,$$

(7.16)

see e.g. [29]. If (7.15) and (7.16) hold, one can omit the $\phi$–integration on the r.s. of (7.11) (which amounts to declaring that const \cdot $\mathcal{D}^2 h \phi = 1$). Eqs. (7.8), (7.15) and (7.16) characterize what one calls (tree-level) critical bosonic string theory. The equations (7.8) and (7.15), for any non-negative value of $c$, mean that the two-dimensional quantum field theory defined by the action $S_{\text{tot}, \Sigma}(X, h)$ in eqs. (7.3–5), for an arbitrary but fixed choice of $\Sigma$ and $\hat{h}$, should be a conformal field theory [69]. A standard argument of quantum field theory says that

$$(-i)^n \frac{\delta^n}{\delta h_{\alpha_1\beta_1}(\xi_1) \ldots \delta h_{\alpha_n\beta_n}(\xi_n)} \ln Z_{\Sigma}(h) = \langle T(T_{\alpha_1\beta_1}(\xi_1) \ldots T_{\alpha_n\beta_n}(\xi_n)) \rangle_{\hat{h}} ,$$

(7.17)

where $\langle T(\cdot) \rangle_{\hat{h}}$ denotes the time ($\tau$)–ordered, connected “vacuum expectation” of the field theory on $(\Sigma, h)$, and $T_{\alpha\beta}(\xi)$ is its energy-momentum tensor at the point $\xi \in \Sigma$. Combining (7.12), (7.15) and (7.17), and setting $T(\xi) = T_{\alpha}(\xi) = h^{\alpha\beta}(\xi) T_{\alpha\beta}(\xi)$ (trace of the energy-momentum tensor) we find that

$$\langle T(\xi) \rangle_{\hat{h}} = - \frac{c}{24 \pi} r_{\hat{h}}(\xi) \bigg|_{\phi=0}$$

(7.18)

and

$$\langle T(T(\xi)T(\eta)) \rangle_{\hat{h}} = - \frac{\delta^2}{\delta \phi(\xi) \delta \phi(\eta)} \ln Z_{\Sigma}(e^{\phi} h) \bigg|_{\phi=0} = 0 \quad \text{for} \quad \xi \neq \eta .$$

(7.19)

Eqs. (7.18) and (7.19) tell us that

$$T(\xi) = \frac{c}{24 \pi} r_{\hat{h}}(\xi) \mathbf{1}$$

(7.20)

(which vanishes if $\hat{h}$ is flat). Eq. (7.20) is precisely the condition for the field theory on $(\Sigma, \hat{h})$ to be conformal. In a renormalization group analysis of Lagrangian field theory, equation (7.20) can be translated into the condition that the renormalization group $\beta$–function vanish. This condition yields equations for the tensor fields $g_{\mu\nu}$ (metric), $B_{\mu\nu}$ (2-form) and $\Phi$ (dilaton) on the (target) space-time manifold $M$. These equations are generalizations of Einstein’s equations (see Section 1). They are quite complicated; see [29,70]. When space-time $M$ is static then they approximately look as follows: In local
coordinates $X^\mu$ on $M$ with the property that $g_{0j}(X) = 0$, $g_{00}(X) = -1$, and choosing a gauge such that $B_{0j}(X) = 0$, $j = 1, \ldots, d - 1$,

$$R_{ij} + 2 \nabla_i \nabla_j \Phi - \frac{1}{4} H_{imn} H_{jn}^{mn} = 0,$$

$$- \frac{1}{2} \nabla_m H_{ij}^m + H_{ij}^m \phi_m = 0,$$

$$C^{(d)} - 26 = 0,$$

where $\nabla$ is the Levi-Civita connection on $M$, $R_{ij}$ is the Ricci tensor, $H_{ijk} = 3 \partial_i (B_{jk})$, and

$$C^{(d)} = d - \frac{3}{2} \alpha' \left[ r - \frac{1}{12} H_{ijk} H^{ijk} - 4 \nabla^i \Phi \nabla_j \Phi + 4 \triangle_g \Phi \right],$$

where $r$ is the scalar curvature on $M$. Eqs. (7.21) hold to one-loop order in the expansion parameter $\alpha'$. (We recall that it is assumed, here, that $g$, $B$ and $\Phi$ are time-independent.) Eqs. (7.21) are the Euler-Lagrange equations corresponding to the action

$$S^{(d)}(g, B, \Phi) = \int d^d X \sqrt{g(X)} e^{-2\Phi(X)} \left[ C^{(d)}(X) - 26 \right].$$

Recalling expression (7.22) for $C^{(d)}$, we observe that this is a generalization of the Hilbert-Einstein action with a cosmological constant $\propto d - 26$. The vanishing of the cosmological constant then requires that the dimension $d$ of space-time should be 26.

Of course, it is of interest to generalize eqs. (7.21–23) to non-static space-times; for results see [70].

Physicists are intrigued by the chain of arguments leading from (7.15) to (7.23), and they have discovered a number of different ways to reach these conclusions; see [29]. They are even more intrigued by the observation that string theory automatically describes interactions (scattering) between different strings, and that one does not have to talk about (target) space-time $M$ explicitly, in order to describe those interactions (in contrast to point-particle field theory): In order to calculate the connected part of a scattering amplitude from $n$ incoming to $m$ outgoing strings, one generalizes expression (7.7) to surfaces $\Sigma$ with $n$ positively and $m$ negatively oriented boundary components, and one sums over all possible topologies (and integrates over the moduli space of conformal structures) of $\Sigma$. As there is no nice theory of Lorentzian surfaces of higher genus and with many boundary components, one performs a Wick rotation, $\xi \equiv (\sigma, \tau) \to (\sigma, i \tau)$, with the effect that the surfaces $\Sigma$ become Riemann surfaces. The different terms in the sum over topologies are then weighted by factors

$$\exp \left( -\text{const} \langle \Phi \rangle H(\Sigma) \right),$$

where $H(\Sigma)$ is the number of handles of $\Sigma$ and $\langle \Phi \rangle$ is some mean value of the dilaton field $\Phi$. Eq. (7.24) follows from (7.5) (with $\Phi$ replaced by $\langle \Phi \rangle$) and the Gauss-Bonnet formula. It is the number $\exp \left( -\text{const} \langle \Phi \rangle \right)$ that is a measure for the deformation parameter, mentioned in Section 3, of the deformation from classical to quantum space-time geometry.

Of course, it is difficult to calculate the various contributions, e.g. to (7.7), in an expansion in the number of handles of $\Sigma$, and the expansion has been argued to be neither convergent nor Borel summable [71]. There are good reasons for these problems:
First, critical bosonic string theory is really a sick theory. When one calculates all the modes of a string propagating in an (approximately) flat space-time $M$, using (7.11–16), one finds that among these modes there is a tachyon with negative mass $^2 = -(\alpha')^{-1}$. This is physically unacceptable, but the problem is cured by replacing the bosonic string by the superstring and performing the GSO projection — see e.g. [29]. But, second, one finds a tower of modes with mass $^2 = (\alpha')^{-1}n$, $n = 0, 1, 2, \ldots$. It is plausible that $\alpha' \propto l_P^2$, where $l_P$ is the Planck length (see Section 3). Thus, for large $n$, a string mode has a mass that can be considerably larger than the Planck mass! Exciting such a mode (and letting it interact with other strings) ought to produce a major perturbation in the geometry (and, perhaps, the topology) of space-time $(M, g)$. However, $(M, g)$ is treated as a fixed classical background space-time in string perturbation theory. Thus we are bound to run into problems with the traditional approach to string perturbation theory; and the superstring is no better than the bosonic string, in this respect!

Let us try to make this a little more precise: Exciting a string mode in a local region of space-time must perturb space-time geometry in a neighborhood of that region. This “back reaction” can be interpreted as a coherent excitation of massless modes, such as gravitons, of an arbitrary number of further strings. It really just does not make sense, ultimately, to talk of some finite number of excited strings propagating through space-time and to describe them as if they were individual particles in a conventional quantum field theory on a fixed space-time manifold $(M, g)$. Because of graviton emission and absorption — which should really be treated non-perturbatively — the very concept of a single particle (or of a finite number of particles) does not make sense in a quantum theory coupling matter to gravitation, and it does not make sense to treat a single particle as a quantum-mechanical subsystem. Likewise, it cannot make sense to talk about a finite number of strings propagating through space-time — one must search for a non-perturbative definition of string theory.

Of course, the problem of the gravitational interactions of very massive string modes should be cured, ultimately, by the feature that space-time has a quantum structure at very small scales and that very massive modes cannot be localized in very tiny space-time regions — one of the reasons for introducing string theory — as described in Section 3, (3.18–21). In fact, it can be argued [71] that string theory predicts uncertainty relations of the kind $\Delta x \geq \frac{1}{\Delta p} + \alpha' \Delta p$, which imply (3.18) when $\alpha' \approx l_P^2$.

All we can really hope to learn from the present naive formulation of string theory is what it might tell us about “string vacua”, i.e., tree-level ($\langle \Phi \rangle \to \infty$) solutions of string theory describing some kind of static space-time filled with static matter fields in which no events take place (but which might not be a classical manifold but some non-commutative space with the property that “sub-manifolds” of certain dimensions, “branes”, have fuzzy loci as a consequence of string zero-point oscillations).

Let us briefly return to eq. (7.11) for the string propagator. Of course, it may happen that the integrand does depend on the field (the Liouville mode) $\phi$. One then speaks of non-critical string theory. Non-critical string theory is not particularly well understood. What has been studied in some detail are models leading to a functional $Z_{\Sigma}(\hat{h})$ that satisfies (7.8) and

$$Z_{\Sigma}(e^{i\phi} \hat{h}) \approx e^{ic_{\Gamma_{\hat{h}}(\phi)}} Z_{\Sigma}(\hat{h}), \quad (7.25)$$

*This infrared problem is analogous to, but much worse than the one familiar from quantum electrodynamics.
i.e., models which are small perturbations of conformal field theories, for arbitrary values of \( c \). One then appears to find that either \( c \leq 1 \) or \( c \geq 25 \), otherwise the theory is inconsistent \([72]\). (However, in \([73,74]\) it is argued that there are other, in particular discrete values of \( c \in (0,25) \) for which the theory can be defined). Let us consider a model with \( c = 25 \), and equality in (7.25), and let us assume that e.g.

\[
Z_{\Sigma}(h) = \int \mathcal{D}h \, X \, e^{i S_{\text{tot.},\Sigma}(X,h)}, \tag{7.26}
\]

with \( S_{\text{tot.},\Sigma}(X,h) \) as in (7.3)–(7.5). Furthermore, we assume that \((M,g)\) is a twenty-five dimensional Riemannian manifold, and that the fields \( g_{\mu\nu}, B_{\mu\nu} \) and \( \Phi \) satisfy eqs. (7.21) — more precisely, the equations expressing that the renormalization group \( \beta \)–function vanishes. In this situation, one can identify \( M \) with physical space, space-time being equal to \( N := M \times \mathbb{R} \), coordinate functions on \( N \) are given by

\[
X^0 = \text{const} \cdot \sqrt{\alpha'} \phi \quad \text{and} \quad X^\mu, \mu = 1, \ldots, 25, \quad \tag{7.27}
\]

and the metric \((g_{\mu\nu})\) on \( N \) is given by

\[
g_{00} \equiv -1, \quad g_{0\mu} \equiv 0, \quad \mu = 1, \ldots, 25, \quad \tag{7.27}
\]

and \((g_{\mu\nu})\), \( \mu, \nu = 1, \ldots, 25 \), is the metric on \( M \). This interpretation is consistent with eqs. (7.11,12) (with \( r_h = 0 \)) and (7.15) (for \( c = 25 \)); note that the sign of \( g_{00} \) follows from the equation \( c - 26 = -1 \). Thus, the Liouville mode \( \phi \) appears as the time coordinate on (a static) space-time \( N = M \times \mathbb{R} \). We leave it open to decide whether there is something profound about this observation; see \([72]\).

Another approach to calculating the Feynman propagator \( \triangle_F(X_i, X_f) \), eqs. (7.6,7), is to discretize the surface \( \Sigma \) (e.g. one replaces \( \Sigma \) by the vertices, edges and faces of a triangulation of \( \Sigma \)) and to interpret \( \mathcal{D}h \) as a sum over all isomorphism classes of triangulations of \( \Sigma \); see \([75]\) and refs. given there. Finally, in accordance with the general philosophy of these notes, one can replace \( \Sigma \) (and thus \( M \)) by a non-commutative space, e.g. the non-commutative torus \([76]\). These last two approaches offer some chance that one will be able to sum over different topologies of \( \Sigma \).

What has remained conspicuously vague in our discussion is what the right interpretation of the arguments \( X_i, X_f \) in the Feynman string propagator \( \triangle_F(X_i, X_f) \) of eqs. (7.6,7) is. In quantum field theory of scalar point-particles, the arguments \( x \) and \( y \) of the Feynman propagator \( \triangle_F(x, y) \) in eq. (7.2) are points in physical space-time, perhaps augmented by internal degrees of freedom; and, rather than defining \( \triangle_F(x, y) \) by (7.2), it can be defined as the time-ordered solution of the Schwinger-Dyson equation

\[
(\Box_g + \mu^2) \, \triangle_F(x, y) = \text{const} \cdot l^{-1} \delta_y^{(d)}(x); \tag{7.28}
\]

\( \delta_y^{(d)} \) is the \( d \)–dimensional \( \delta \)–function on \( M \) concentrated at \( y \).

String theory, being intended to be a theory of quantum gravity, should not be formulated in a way that refers to any specific choice of a target space-time \( M \) (“background independence”). Thus, we are actually not supposed to think of \( X_i \) and \( X_f \) as some un-parametrized loops embedded in some specific target space-time \( M \). They really should just represent unparametrized loops, decorated by “internal degrees of freedom” intrinsic to the string, but not referring to a specific model of target space-time \( M \). Surprisingly,
this remark suggests a fairly concrete analogue of (7.28) as a stringy Schwinger-Dyson equation for $\Delta_F(X_i, X_f)$. This is the theme of the next section, where we shall draw on material from Sects. 4.2 and 5.3.

## 7.2 A Schwinger-Dyson equation for string Green functions from reparametrization invariance and world-sheet supersymmetry

The Feynman propagator of scalar free field theory on a space-time $(M, g)$ is a solution of the equation

$$ (\Box_g + \mu^2) \Delta_F(x, y) = \text{const} \cdot l^{-1} \delta^{(d)}(x) . $$

Here $\Box_g$, the d’Alembertian on $(M, g)$, is a hyperbolic operator. There is no natural Hilbert space to which $\Delta_F$ belongs. Rather, $\Delta_F$ belongs to some space of distributions which is a module for some algebra of hyperbolic differential operators. Concepts from the theory of (self-adjoint, normal, . . . ) operators on Hilbert space are, a priori, a little out of place in attempting to solve (7.28).

Suppose, however, that $(M, g)$ is a product space $(M, g) = (N, \eta) \times (L, G)$, (7.29) where $(N, \eta)$ is a $(d-n)$–dimensional Lorentzian manifold, and $(L, G)$ is an $n$–dimensional (e.g. compact) Riemannian manifold; for example, set $d = 4, n = 2, N = \mathbb{M}^2$ (two-dimensional Minkowski space), $L = \text{disk} \subset \mathbb{R}^2$, and think of a wave guide filled with a scalar field. Then (7.28) can be solved by separation of variables, and we must study the eigenvalues and eigenfunctions of the Laplace-Beltrami operator $-\Delta_G$ which defines a positive, self-adjoint operator densely defined on the Hilbert space $L^2(L, dvol_G)$. Now, this one is a problem in the theory of operators on Hilbert space; and once it is solved, the problem of solving (7.28) is reduced to a hyperbolic problem on a space of distributions on $(N, \eta)$ — as we have learnt in school.

If we are interested in the Feynman propagator for a free field theory of particles with spin transforming under the spinor representation of Spin$(d-1,1)$, eq. (7.28) is replaced by

$$ (D + \mu) S_F(x, y) = \text{const} \cdot \delta^{(d)}(x) , $$

(7.30)

where $D$ is what legitimately is called Dirac operator (as opposed to the “Pauli-Dirac operator” of Section 4), which is a hyperbolic differential operator acting on a space of distributional sections of the spinor bundle over $(M, g)$. If $(M, g)$ is of the form (7.29), eq. (7.30) can be solved by separation of variables: If $D^N$ denotes the hyperbolic Dirac operator acting on distributional sections of the spinor bundle over $(N, \eta)$ and $D^L$ denotes the elliptic Pauli-Dirac operator acting on smooth sections of the spinor bundle over $(L, G)$, then

$$ D = D^N \otimes 1 + \gamma \otimes D^L , $$

(7.31)

where $\gamma$ is a $\mathbb{Z}_2$–grading for $D^N$, and $D$ acts on the tensor product of the two spaces of sections. The solution of (7.30) involves studying the spectrum and the eigenfunctions of $D^L$, which is a self-adjoint operator defined on a dense domain in the Hilbert space $\mathcal{H}_e$ of square-integrable sections of the spinor bundle over $(L, G)$, as discussed in Section 4.
Again, we encounter a problem in the theory of operators on Hilbert space. It involves \( N = 1 \) supersymmetric spectral data \( (A = C(L), \mathcal{H}_e, DL) \).

We could also study a free field theory of particles with spin described by fields which, classically, are differential forms over \((M, g)\). The calculation of the Feynman propagator then involves two hyperbolic Dirac operators, \( D \) and \( \bar{D} \), and, in the situation described in (7.29), this problem requires the study of \( N = (1, 1) \) supersymmetric spectral data, \( (A = C(L), \mathcal{H}_{e-p}, DL, \bar{DL}, \gamma, \bar{\gamma}) \), as considered in Sects. 4 and 5.2. Then we have

\[
D = DN \otimes 1 + \gamma \otimes DL, \\
\bar{D} = \bar{DN} \otimes 1 + \gamma \otimes \bar{DL},
\]

(7.32)

where \( DN, \bar{DN} \) are hyperbolic and \( DL, \bar{DL} \) are elliptic.

An arbitrary Green function \( D_F(x,...) \) of this theory satisfies the equations

\[
D D_F(x,...) = 0 = \overline{D} D_F(x,...),
\]

(7.33)
as long as \( x \) does not coincide with any other argument of \( D_F(x,...) \) and as long as the theory is a theory of free fields.

Incidentally, there are Feynman path integral expressions for the solutions of eqs. (7.30,33); they can be inferred from eqs. (4.47) and (4.48). Their generalizations to bosonic string theory have been discussed, in part, in Sect. 7.2. We ask: What is the generalization of eqs. (7.28), (7.30) and (7.33) to bosonic or spinning (super) string theory, respectively?

We start with the generalization of (7.28), whose solutions should be the propagator \( \triangle_F(X_i, X_f) \) of eq. (7.7). We first consider tree-level string theory \( (\langle \Phi \rangle \to \infty \text{ in } (7.24)) \). According to the discussion following eq. (7.28), we guess that \( \triangle_F(X := X_i,...) \) belongs to some, as yet mysterious, “space of distributions” which is a module, \( S' \), for some, as yet mysterious, “algebra of hyperbolic operators”. This algebra must contain an analogue, \( \Box \), of the d’Alembertian \( \Box_g \), and one of the equations satisfied by \( \triangle_F(X,...) \) must be

\[
\Box \triangle_F(X,...) = 0,
\]

(7.34)
at “non-coinciding arguments”.

We consider closed strings. Then the module \( S' \) should carry a (projective) representation of the Witt algebra \( W := \text{Der } (C(S^1)) \) of vector fields on \( S^1 \) (i.e., of infinitesimal diffeomorphisms of \( S^1 \)), which is interpreted as the algebra of infinitesimal reparameterizations of the string. \( W \) is an infinite-dimensional Lie algebra, whose complexification has a basis \( \{l_n\}_{n \in \mathbb{Z}} \) with structure relations

\[
[l_n, l_m] = (n - m) l_{n+m}.
\]

(7.35)

This Lie algebra has projective representations, which are representations of a central extension of \( W \), called Virasoro algebra, which has a basis \( \{L_n\}_{n \in \mathbb{Z}} \) satisfying the structure relations

\[
[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12} n(n^2 - 1) \delta_{n+m,0},
\]

(7.36)

where \( c \) is the central element. This element is invisible on the subalgebra \( sl_2(\mathbb{R}) \) of those infinitesimal Möbius transformations that leave the unit circle invariant, with basis \( \{L_{-1}, L_0, L_1\} \).
In quantum field theory, $\Delta_F(x, \ldots)$ is also given by

$$\Delta_F(x, \ldots) = \langle T(A(x) \ldots) \rangle \tag{7.37}$$

where $\langle T(\ldots) \rangle$ denotes the time-ordered vacuum expectation value, and $A$, the free scalar field, is an operator-valued distribution on space-time $M$. In analogy to (7.37), one might expect that, in bosonic string theory, there is a free closed-string field $\hat{\mathcal{A}}$ which is an operator-valued distribution on the space $M^{S_1}$ of parametrized loops $X^\mu(\sigma)$ in $M$ ($0 \leq \sigma < 2\pi$) such that

$$\Delta_F(X_i, X_f) = \text{const} \cdot \int_0^\infty dT \left( e^{iT(\square + i0)} \right) (X_i, X_f) \tag{7.38}$$

where $X_i = X_i(\sigma)$ and $X_f = X_f(\sigma)$, $0 \leq \sigma < 2\pi$, are loops in $M^{S_1}$. The first equation is (7.11) with the surface $\Sigma$ being the cylinder

$$\Sigma = \left\{ \xi \equiv (\sigma, \tau) \mid 0 \leq \sigma < 2\pi, \ 0 \leq \tau < \infty \right\} \tag{7.39}$$

Furthermore, $\hat{h} = \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$. The second equation says that $\Delta_F(X_i, X_f)$ is a solution of eq. (7.34) analogous to the solution

$$\Delta_F(x, y) = \text{const} \cdot \int_0^\infty dT \left( e^{iTl^2 + i0} \right) (x, y) \tag{7.40}$$

of eq. (7.28). The third equation says that $\Delta_F(X_i, X_f)$ is the “time-ordered” vacuum expectation value of a string field $\hat{\mathcal{A}}$. (Thus, one should be able to express $\Delta_F(X_i, X_f)$ in terms of matrix elements of a unitary quantum-mechanical string propagator.)

Reparametrization invariance should be the statement that, for “non-coinciding arguments”,

$$\lambda_n \langle T \left( \hat{\mathcal{A}}(X^\mu(\sigma)) \ldots \right) \rangle = \langle T \left( \hat{\mathcal{A}}((l_n X^\mu)(\sigma)) \ldots \right) \rangle + \omega_n \langle T \left( \hat{\mathcal{A}}(X^\mu(\sigma)) \ldots \right) \rangle \equiv 0 \tag{7.41}$$

where the operators $\{ \omega_n \equiv \omega(l_n) \}_{n \in \mathbb{Z}}$ are introduced in order to allow for projective representations: Eq. (7.41) is intended to say that the space $S'$ be a module for the Witt algebra $W$ (then $\omega_n = 0$ for all $n$), or for a central extension of $W$ (i.e., for the Virasoro algebra). Thus the operators $\lambda_n$ representing $l_n$ on $S'$ should satisfy the relations

$$[\lambda_n, \lambda_m] = (n - m)\lambda_{n+m} + \frac{c}{12} n(n^2 - 1) \delta_{n+m,0} \tag{7.42}$$

94
We conclude that $\Delta_F$ is an element of some module $S'$ solving the equations
\[ \Box \tau(X, \ldots) = 0 , \quad \lambda_n \tau(X, \ldots) = 0 , \] (7.43)
for all $n \in \mathbb{Z}$, again at “non-coinciding” arguments. According to our previous discussion in Sects. 4.2, eqs. (4.86–89) and 5.3, eqs. (5.116–125), the problem of solving eqs. (7.43) can be viewed as a problem in BRST cohomology. But, in order to find a nilpotent BRST operator $Q_{\text{BRST}}$ of which $\Delta_F$ will represent a cohomology class, we must first determine the Lie algebra $G_\Box$ generated by $\{ \Box : \lambda_n \}$. Logically, we do not seem to have enough data to find a unique solution for $G_\Box$! But our discussion between (7.38) and (7.43) suggests a solution: If we define
\[ \Delta_{\tau}^{(r)}(x, y) = \text{const} \cdot \int_\tau^\infty d\tau \left( e^{iT(\Box y + \mu^2 + i0)} \right)(x, y) \]
then
\[ -i l^{-1} \frac{d}{d\tau} \Delta_{\tau}^{(r)}(x, y) \bigg|_{\tau=0} = (\Box y + \mu^2)\Delta(x, y) = 0 \]
at non-coinciding arguments. Likewise, we interpret $\Delta_{\tau}^{(r)}(X_i, X_f)$ as a time-ordered Green function of the operator $\Box$, as in eq. (7.38), and define
\[ \Delta_{\tau}^{(r)}(X_i, X_f) = \text{const} \cdot \int_\tau^\infty d\tau \left( e^{iT(\Box + i0)} \right)(X_i, X_f) . \]
Then we find that
\[ -i \frac{d}{d\tau} \Delta_{\tau}^{(r)}(X_i, X_f) \bigg|_{\tau=0} = \Box \Delta_{\tau}^{(r)}(X_i, X_f) \bigg|_{\tau=0} = 0 , \] (7.44)
at “non-coinciding” arguments. The path integral representation of $\Delta_{\tau}^{(r)}(X_i, X_f)$ involves the cylinder
\[ \Sigma_\tau = \{ \xi = (\sigma, \tau') \mid 0 \leq \sigma < 2\pi, \tau \leq \tau' < \infty \} \]
\[ \sim \{ x \in \mathbb{C} \mid e^\tau \leq |z| < \infty \} . \] (7.45)
Thus $\Box$ represents the generator of the dilatation
\[ z \mapsto e^\tau z , \quad \tau > 0 , \] (7.46)
of the complex plane on the space $S'$. We already know that the operators $\lambda_n, n \in \mathbb{Z},$ represent complex vector fields on the circle $\{ e^{i\sigma} \mid 0 \leq \sigma < 2\pi \}$ as operators on $S'$. In particular, since $\lambda_0$ represents the generator of a uniform rotation $e^{i\sigma} \mapsto e^{(i\sigma + \theta)}$, the operator $\frac{1}{2}(\Box + \lambda_0)$ generates translations along the light rays $\{ \tau - \sigma = \text{const.} \}$
\[ \tau + \sigma \mapsto \tau + \sigma + \theta , \quad \tau - \sigma \mapsto \tau - \sigma , \]
while \( \frac{1}{2} (\Box - \lambda_0) \) represents the generator of
\[
\tau + \sigma \mapsto \tau + \sigma, \quad \tau - \sigma \mapsto \tau - \sigma + \theta.
\]
Now we can guess a solution to the problem of determining the Lie algebra \( G \) generated by \( \{\Box, \lambda_n, n \in \mathbb{Z}\} \): There are two Virasoro algebras, Vir and \( \overline{\text{Vir}} \), with bases \( \{L_n\}_{n \in \mathbb{Z}} \) and \( \{\bar{L}_n\}_{n \in \mathbb{Z}} \) such that
\[
\Box = L_0 + \bar{L}_0 + \text{const.}, \quad \lambda_n = L_n - \bar{L}_{-n},
\] (7.47)
and the generators \( L_n^\# \) (which denotes \( L_n \) or \( \bar{L}_n \)) satisfy the relations
\[
[L_n^\#, L_m^\#] = (n - m) L_{n+m}^\# + \frac{\lambda_0}{12} n (n^2 - 1) \delta_{n+m,0}, \quad [L_n, \bar{L}_m] = 0,
\] (7.48)
for all \( n, m \in \mathbb{Z} \). The operators \( L_n \) and \( \bar{L}_n \) represent generators of reparametrizations \( \tau + \sigma \mapsto f_+ \tau + (\tau + \sigma) \) and \( \tau - \sigma \mapsto f_- (\tau - \sigma) \), respectively, on the space \( S' \). Thus, suitable combinations of the operators \( L_n^\#, n \in \mathbb{Z} \), generate the conformal semi-group of maps from \( \{z^\# \mid |z^\#| \geq 1 \} \) into itself. Clearly, the operators \( L_0 = \frac{1}{2} (\Box + \lambda_0 + \text{const.}), \bar{L}_0 = \frac{1}{2} (\Box - \lambda_0 + \text{const.}) \) and \( \{\lambda_n\}_{n \in \mathbb{Z}} \) as in eq. (7.47) provide a representation of \( \text{Vir} \times \overline{\text{Vir}} \) on \( S' \).

Thus, in order to solve the equations (7.43), we must introduce two BRST operators, \( Q_{\text{BRST}} \) and \( \overline{Q}_{\text{BRST}} \), whose form is determined by comparing formulas (4.68), (4.86) and (7.48):
\[
Q_{\text{BRST}}^\# = \sum_{n \in \mathbb{Z}} c_n^\# L_n^\# - \frac{1}{2} \sum_{n, m \in \mathbb{Z}} (n - m) c_n^\# c_m^\# b_{n+m}^\#: - a^\# c_0^#,
\] (7.49)
where the double colons denote standard Wick ordering (move operators with index \( n > 0 \) to the right of operators with index \( m < 0 \), using anti-commutativity), and \( a^\# \) is a constant arising from an ambiguity in the definition of Wick ordering [29]. The operator \( T^\# \) determining the degree of differential forms is given by
\[
T^\# = \frac{1}{2} (c_0 b_0 - b_0 c_0) + \sum_{n=1}^{\infty} (c_{-n} b_n - b_{-n} c_n).
\] (7.50)
Since Vir is an infinite-dimensional Lie algebra and because of Wick ordering ambiguities, it is not automatic that
\[
(Q_{\text{BRST}}^\#)^2 = 0.
\] (7.51)
The condition for eq. (7.51) to hold turns out to be
\[
c^\# = 26, \quad a^\# = 1,
\] (7.52)
see e.g. [29], and compare to (7.11,12). The solution \( \Delta_F (X_i, \cdot) \) of eqs. (7.43) must be a cohomology class of the double complex
\[
\left( S' \otimes \Lambda(\text{Vir}^*) \otimes \Lambda(\overline{\text{Vir}}^*) ; Q_{\text{BRST}}, \bar{Q}_{\text{BRST}} \right)
\]
and, upon closer examination (see [29]), it must have degree \((-\frac{1}{2}, -\frac{1}{2})\), i.e.

\[
T \triangle_F(X_i, X_f) = \overline{T} \triangle_F(X_i, X_f) = -\frac{1}{2} \triangle_F(X_i, X_f).
\] (7.53)

Physicists call the eigenvalues of $T$ and $\overline{T}$ “ghost numbers”. It is not very difficult to determine the cohomology classes of $Q_{BRST}$ and $\overline{Q}_{BRST}$ of ghost number \((-\frac{1}{2}, -\frac{1}{2})\), see [29]: As a functional of $X_i$, $\triangle_F(X_i, \cdot)$ must be a solution of the equations

\[
ce_n^\# \tau(X_i, \ldots) = b_n^\# \tau(X_i, \ldots) = 0, \quad \text{for all } n > 0
\]

and

\[
(L_n^\# - \delta_n,0) \tau (X_i, \ldots) = 0, \quad \text{for all } n \geq 0,
\]

where $\tau(X_i, \ldots) \in \mathcal{S}' \otimes \Lambda(\text{Vir}^*) \otimes \Lambda(\overline{\text{Vir}}^*)$. Likewise, we must have that

\[
\tau (\ldots, X_f) c_n^\# = \tau (\ldots, X_f) b_{n+1}^\# = 0 \quad \text{for all } n < 0
\]

and

\[
\tau (\ldots, X_f)(L_n^\# - \delta_n,0) = 0, \quad \text{for all } n \leq 0.
\]

From these equations one can derive that eqs. (7.43) are satisfied by $\triangle_F(X_i, X_f)$, at non-coinciding arguments, with $\square$ and $\lambda_n$ as in eqs. (7.47). If one insists that these equations hold for all arguments the solution is not a string propagator, but it would be a two-string Wightman distribution. Green functions of interacting string theories are expected to be solutions of inhomogeneous versions of eqs. (7.43); see [80].

The data $(\mathcal{S}', \square, G_{\square})$ are analogous to spectral data $(\mathcal{A}, \mathcal{H}, \triangle)$ of non-commutative metric spaces, as described in point (1) of the introduction to Section 5 and generalized in Sect. 5.3. But there are important differences: The module $\mathcal{S}'$ for $G_{\square}$ is a space of distributions and is not equipped with a positive semi-definite inner product, while $\mathcal{H}$ is a Hilbert space; the operator $\square$ is hyperbolic, while $\triangle$ is elliptic. Moreover, in the data $(\mathcal{S}', \square, G_{\square})$, we have not specified an algebra $\mathcal{A}$, yet, on which the Witt- or Virasoro algebra acts as an algebra of infinitesimal reparametrizations.

At this point, we should recall our brief discussion of separation of variables after eq. (7.29) and in (7.31,32). In analogy to that discussion, we propose the following definition:

We say that, in solving eqs. (7.43) for the string propagator $\triangle_F(X_i, X_f)$, one can separate variables iff

\[
\square = L_0 + \overline{L}_0, \quad \lambda_n = L_n - \overline{L}_{-n}, \quad n \in \mathbb{Z},
\] (7.54)

with

\[
L_n^\# = L_n^\# e \otimes 1 + 1 \otimes L_n^\# i
\]
(7.55)
(where $L_n^\#$ denotes $L_n$ or $\bar{L}_n$). The sets $\{L_n^\#\}_{n \in \mathbb{Z}}$ and $\{\bar{L}_n^\#\}_{n \in \mathbb{Z}}$ span commuting Virasoro algebras $\text{Vir}^{e,i}$ and $\overline{\text{Vir}}^{e,i}$ with central charges $c^e, c^i$ and $\bar{c}^e, \bar{c}^i$, respectively, such that

\begin{equation}
(i) \quad c^e + c^i = 26, \quad \bar{c}^e + \bar{c}^i = 26, \quad (7.56)
\end{equation}

and

\begin{equation}
(ii) \quad \text{the commuting Virasoro algebras } \text{Vir}^i \text{ and } \overline{\text{Vir}}^i \text{ are unitarily represented on a Hilbert space } \mathcal{H}^i, \text{ i.e.,}
\end{equation}

\begin{equation}
(L_n^i)^* = L_{-n}^i, \quad (\bar{L}_n^i)^* = \bar{L}_{-n}^i, \quad (7.57)
\end{equation}

for all $n \in \mathbb{Z}$, where * is the adjoint for operators on the Hilbert space $\mathcal{H}^i$.

One usually also requires that

\begin{equation}
c^i = \bar{c}^i. \quad (7.58)
\end{equation}

It follows from (7.57) that $c^i, \bar{c}^i \geq \frac{1}{2}$ and that $L_0$ and $\bar{L}_0$ are positive operators on $\mathcal{H}^i$.

The module $S^\prime$ is then a tensor product, $S^\prime = S^e \otimes \mathcal{H}^i$, and the solution of (7.43), more precisely of the equations

\begin{equation}
\overline{\text{BRST}} Q^\# \tau(X, \ldots) = 0, \quad \overline{T}^\# \tau(X, \ldots) = -\frac{1}{2} \tau(X, \ldots), \quad (7.59)
\end{equation}

requires the study of the unitary representations of $\text{Vir}^i$ and $\overline{\text{Vir}}^i$ on $\mathcal{H}^i$ and, in particular, of the spectrum and the eigenvectors of $L_0$ and $\bar{L}_0$.

The data $(\mathcal{H}^i, \{L_n^\#\}_{n \in \mathbb{Z}})$ could come from a unitary conformal field theory, as discussed in the next section. In this case, the mathematical problem to be studied is to understand in how far a unitary conformal field theory determines a (generally non-commutative) Riemannian space $(L, G)$ describing the geometry of “internal degrees of freedom” of a tree-level string theory. This is the problem addressed in refs. [78,24]. Thus, apparently, unitary conformal field theories take the place of the spectral data

\begin{equation}
(A := C(L), \quad \mathcal{H} := L^2(L, dvol_G), \quad \Delta = -\Delta_G),
\end{equation}

which appear in the solution of the Schwinger-Dyson equation (7.28) in the situation described in (7.29).

If the Virasoro algebras $\text{Vir}^e$ and $\overline{\text{Vir}}^e$ describe string propagation in an “external” Minkowski space $(N, \eta)$ and if the string theory is non-chiral (left- and right moving sectors are isomorphic) then eqs. (7.43) and (7.59) imply that the mass $m$ of a string mode is given by the formula

\begin{equation}
m^2 = h + \bar{h} + n - 2, \quad (7.60)
\end{equation}

where $h^\# \in \text{spec } L^\#_0$ and $n = 0, 1, 2, \ldots$; the contribution $-2$ on the r.s. of (7.60) comes from the equation $a^\# = 1$, see (7.49,52). Unfortunately, because of this $-2$, it could happen that $m^2 < 0$, i.e., a tachyon appears. This problem is eliminated in superstring theory. In fact, the analysis of tree-level bosonic string theory just presented can be extended to superstring theory.

A string theory with $N = 1$ supersymmetric data $(S^\prime, D, G_D)$, generalizing the free field theory of particles with spin as considered in eq. (7.30) is heterotic string theory. We refer the reader to [29] and references given there for details. On the Ramond sector of this
theory, one identifies the generalized Dirac operator $D$ with e.g. a left-moving Ramond generator $G_0$, and sets
\[ L_0 := D^2 + \frac{c}{24}, \]  
(7.61)
for some constant $c$ which will turn out to be the central charge of a super-Virasoro algebra. If $\{\lambda_n\}_{n \in \mathbb{Z}}$ are the operators representing reparametrizations of the parameter space $S_1$ of a closed string on the module $S'$ (see eq. (7.41)), one sets, for arbitrary $n \in \mathbb{Z}$,
\[ [\lambda_n, D] \equiv [\lambda_n, G_0] =: \frac{n}{2} G_n, \]  
(7.62)
and
\[ \{G_n, G_m\} =: 2 L_{n+m} + \frac{c}{3} \left( n^2 - \frac{1}{4} \right) \delta_{n+m,0}. \]  
(7.63)
One then demands (or, under suitable hypotheses, proves) that the operators $\{G_n, L_n\}_{n \in \mathbb{Z}}$ obey the additional commutation relations
\[ [L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} n \left( n^2 - 1 \right) \delta_{n+m,0}, \]
\[ [L_n, G_m] = \left( \frac{n}{2} - m \right) G_{n+m}, \]  
(7.63)
which together with (7.62) define the super-Virasoro algebra $s\text{Vir}$. Finally one introduces $\bar{L}_n = L_{-n} - \lambda_{-n}$, and it follows from (7.42) and (7.63) that $\{\bar{L}_n\}$ generate a second Virasoro algebra $\overline{\text{Vir}}$ with some central charge $\bar{c}$. Thus
\[ G_D = s\text{Vir} \times \overline{\text{Vir}}. \]  
(7.64)
The propagator $S_F(X_i, X_f)$ of tree-level heterotic string theory is a solution of the equations
\[ L_n \tau(X, \ldots) = 0, \quad G_n \tau(X, \ldots) = 0, \quad \bar{L}_n \tau(X, \ldots) = 0, \]  
(7.65)
for all $n \in \mathbb{Z}$, at “non-coinciding arguments”. As outlined in Sect. 5.3, the problem of solving eqs. (7.65) can be reformulated as a problem in BRST cohomology. Let $\text{Vir} \equiv s\text{Vir}_{\text{even}}$ be the Virasoro subalgebra contained in $s\text{Vir}$ spanned by $\{L_n\}_{n \in \mathbb{Z}}$, and let $s\text{Vir}_{\text{odd}}$ be the subspace of $s\text{Vir}$ spanned by $\{G_n\}_{n \in \mathbb{Z}}$. We consider the module
\[ S' := S' \otimes \Lambda((s\text{Vir}_{\text{even}})^*) \otimes S((s\text{Vir}_{\text{odd}})^*) \otimes \Lambda(\overline{\text{Vir}}^*). \]  
(7.66)
Here $\Lambda((s\text{Vir}_{\text{even}})^*)$ and $\Lambda(\overline{\text{Vir}}^*)$ are anti-symmetric Fock spaces carrying the Fock representation of the canonical anti-commutation relations
\[ \{c_n, c_m^\#\} = \left\{b_n^\#, b_m^\#\right\} = 0, \quad \{c_n^\#, b_m^\#\} = \delta_{n+m,0}, \]  
(7.67)
and $S((s\text{Vir}_{\text{odd}})^*)$ is a symmetric Fock space carrying the Fock representation of the canonical commutation relations
\[ [\gamma_n, \gamma_m] = [\beta_n, \beta_m] = 0, \quad [\beta_n, \gamma_m] = \delta_{n+m,0}. \]  
(7.68)
Furthermore, $c_n, b_n$ anti-commute with $\bar{c}_m, \bar{b}_m$, and $\gamma_n, \beta_n$ commute with $c_m, \bar{c}_m, b_m$ and $\bar{b}_m$. We define the BRST operator

$$Q_{\text{BRST}} = \sum_n c_n L_n - \frac{1}{2} \sum_{n,m} (n-m) : c_{-n} c_{m+n} : - a c_0$$

$$+ \sum_{n,m} \left( \frac{3}{2} n + m \right) : c_{-n} \beta_{-m} \gamma_{n+m} :$$

$$+ \sum_n \gamma_n G_{-n} - \sum_{n,m} \gamma_{-n} \gamma_{-m} b_{n+m} ,$$

see eq. (5.123). An operator $\bar{Q}_{\text{BRST}}$ is defined as in (7.49). Then one can prove (see [29] and references given there) that

$$Q_{\text{BRST}}^2 = 0 \iff c = 15 \text{ and } a = 0 ,$$

$$\bar{Q}_{\text{BRST}}^2 = 0 \iff \bar{c} = 26 \text{ and } \bar{a} = 1 ,$$

see (7.52). The space $S'$ of eq. (7.66) is a $\mathbb{Z} \times \mathbb{Z}$ graded double complex for $(Q_{\text{BRST}}, \bar{Q}_{\text{BRST}})$, and the string propagator $S_F(X_i, X_f)$ can be characterized as a cohomology class of $(Q_{\text{BRST}}, \bar{Q}_{\text{BRST}})$ of “ghost number” $\left( - \frac{1}{2}, - \frac{1}{2} \right)$; compare also to the discussion following eq. (7.52).

It is well known [29] that heterotic string theory has a second sector, the Neveu-Schwarz sector. There, the spectral data have the form $(S'_{NS}, Q, Q^+, G_{Q,Q^+})$ with

$$\{ Q, Q^+ \} = 2 L_0 ,$$

and $G$ is still a Virasoro algebra providing a projective representation of infinitesimal reparametrizations of $S^1$. One sets $Q =: G_{1/2}, Q^+ =: G_{-1/2}$, and defines

$$[\lambda_n, G_{1/2}] =: \frac{n-1}{2} G_{n+1/2} , \ n \in \mathbb{Z} ,$$

$$\{ G_n, G_m \} = 2 L_{n+m} + \frac{c}{3} \left( n^2 - \frac{1}{4} \right) \delta_{n+m,0} .$$

(7.72)

The algebra generated by $\{ G_{n+1/2}, L_n \}_{n \in \mathbb{Z}}$ is again characterized by the relations given in (7.62,63), but the operators $G_n$ now have labels $n \in \mathbb{Z} + \frac{1}{2}$.

In the Ramond sector, with $S' =: S_R$, the left-moving fermionic string modes have periodic boundary conditions on parameter space $S^1$, while in the Neveu-Schwarz sector, with $S' =: S'_{NS}$, they have anti-periodic boundary conditions on $S^1$. In the context of string theory, the existence of the Neveu-Schwarz sector and the disappearance of tachyons from the spectrum of modes of heterotic string theory follow from the condition that the amplitude for closed string propagation along a toroidal world-sheet be modular invariant.

From our discussion in (7.31) and (7.54–58) one can guess how to define a notion of separation of variables in heterotic string theory. Separation of variables leads to the consideration of spectral data $(\mathcal{H}^i, \{ L^i_n \}_{n \in \mathbb{Z}})$ defining a unitary conformal field theory.
But, in contrast to purely bosonic string theory, the conformal field theories encountered in the study of heterotic string theory typically have supersymmetries: In addition to the Virasoro generators $L_n, \bar{L}_n, n \in \mathbb{Z}$, there are Ramond (and Neveu-Schwarz) generators $G_n, \bar{G}_n$ with $n \in \mathbb{Z}$ (resp. $n \in \mathbb{Z} + 1/2$), and unitarity is the constraint that $(G_n^\#)^*=G_{-n}^\#$ on the Hilbert space $\mathcal{H}$. The formula corresponding to eq. (7.60) is

$$m^2 = h + \bar{h} + n - 1, \quad n = 0, 1, 2, \ldots,$$

where $h^\#$ is an eigenvalue of $L_0^\#$. Thus, in order to identify the massless string modes corresponding to particles like gravitons, gluons, photons, light fermions, we must study the eigenspaces of $L_0^\#$ and $\bar{L}_0^\#$ corresponding to the eigenvalue 0 and $1/2$.

In Sect. 4.1 we have studied Pauli’s non-relativistic quantum theory of an electron and a positron. Heterotic string theory is the stringy analogue of Pauli’s electron. The role of the electromagnetic $U(1)$–connection $A$ in the quantum theory of Pauli’s electron is played by string modes transforming under a gauge group $G = SO(32)$ or $E_8 \times E_8$; see [29]. The gauge symmetry appears in the study of the right-moving modes (with infinitesimal reparametrizations represented by the generators $\bar{L}_n$) which contain a Kac-Moody current algebra at level 1 based on the group $G$. The analogue of Pauli’s positron is a heterotic string theory with reversed roles of left- and right-moving modes. In Sect. 4.1, we also studied the quantum theory of bound states, positronium, of an electron and a positron. This provided us with spectral data $(A, \mathcal{H}_{\text{e-p}}, D, \bar{D})$ displaying $N = (1, 1)$ supersymmetry, see eqs. (4.30–35), from which de Rham-Hodge theory and Riemannian geometry of a classical manifold could be reconstructed. A more general analysis of the passage from $N = 1$ supersymmetric spectral data (electron and positron – spin geometry) to $N = (1, 1)$ supersymmetric spectral data (positronium – Riemannian geometry) has been sketched in subsection 5 of Sect. 5.2. That analysis suggests that it should be possible to construct closed string theories with $N = (1, 1)$ supersymmetric data $(S_R, D, \bar{D}, G_D, \bar{G_D})$ (and with corresponding Neveu-Schwarz data) from two copies of heterotic string theory with $N = 1$ supersymmetric data.

There are two such theories with $N = (1, 1)$ supersymmetry, the type IIA and the type IIB string theories, distinguished from each other by different combinations of left- and right-moving modes describing chiral fermions; see [29] and refs. given there. In these string theories,

$$G_{D, \bar{D}} = s\text{Vir} \times s\overline{\text{Vir}},$$

where the super-Virasoro algebras have generators $\{L_n^\#, G_n^\#\}_{n \in \mathbb{Z}}$ that satisfy the relations (7.62,63), with $D = G_0$ and $\bar{D} = \bar{G}_0$.

We are interested in calculating tree-level string Green functions $D_F(X, \ldots)$ which are solutions of the equations

$$\lambda_n \tau(X, \ldots) = 0, \quad \bar{D}^\# \tau(X, \ldots) = 0$$

(at “non-coinciding” arguments). Under suitable hypotheses, these equations imply the following more precise ones:

$$L_n^\# \tau(X, \ldots) = 0, \quad G_m^\# \tau(X, \ldots) = 0$$

(7.75)
for all \( n \in \mathbb{Z} \), \( m \in \mathbb{Z} \left( \pm \frac{1}{2} \right) \) and where \( \tau \in S'_{R} \) (resp. \( \tau \in S'_{NS} \)).

It is of interest to study solutions of a system of weaker equations. We define a differential \( d \) by setting
\[
d := G_0 - i \bar{G}_0 .
\]
The operators \( d \) and \( d^* := G_0 + i \bar{G}_0 \) can be interpreted as the exterior derivative and its adjoint of a centrally extended \( N = (1,1) \) supersymmetry algebra; see subsection 8) of Sect. 5.2. In addition, we define
\[
d_n := [\lambda_n, d] = G_n - i \bar{G}_n .
\]
In type-II string theories, the central charges \( c \) and \( \bar{c} \) of the left- and the right-moving super-Virasoro algebras coincide (\( c = \bar{c} = 15 \)). Identifying \( \lambda_n \) with \( L_n - \bar{L}_n \) as in eq. (7.47), it follows that the \( \lambda_n \) satisfy the Witt algebra
\[
[\lambda_n, \lambda_m] = (n - m) \lambda_{n+m} ,
\]
see eq. (7.35); furthermore,
\[
[\lambda_n, d_m] = \left( \frac{n}{2} - m \right) d_{n+m} ,
\]
and
\[
\{d_n, d_m\} = 2 \lambda_{n+m} . \tag{7.76}
\]
These commutation relations define the “super-Witt algebra”.

One may argue that equations (7.75) for the Green functions \( D_{F}(X,\ldots) \) of type-II string theory are really more restrictive than they ought to be. The correct general equations for the Green functions \( D_{F}(X,\ldots) \) with Ramond-Ramond boundary conditions (at “non-coinciding arguments”) are the weaker equations
\[
\lambda_n \tau(X,\ldots) = 0 , \quad d_n \tau(X,\ldots) = 0 , \tag{7.77}
\]
for all \( n \in \mathbb{Z} \). It follows from the structure relations of the super-Witt algebra that solutions of eqs. (7.77) are cohomology classes of the operator \( d \), which is nilpotent on the subspace of the module \( S'_{R} \) annihilated by \( \{\lambda_n\}_{n \in \mathbb{Z}} \). Among solutions of eqs. (7.77) one would expect to find ones describing type-II string “solitons” — see also [85].

In attempting to solve eqs. (7.75) for the (tree-level) Green functions of type-II string theory by separation of variables, one is led to studying \( N = (1,1) \) supersymmetric spectral data
\[
\left( \mathcal{H}^i, \{L^i_n\}_{n \in \mathbb{Z}}, \{G^i_m\}_{m \in \mathbb{Z}(+1/2)} \right) \tag{7.78}
\]
derived from an “internal” \( N = (1,1) \) supersymmetric, unitary conformal field theory, as briefly studied in Sect. 7.4. The mass formula (7.73) continues to hold, with \( h \) and \( \bar{h} \) eigenvalues of \( L^i_0 \) and \( \bar{L}^i_0 \), respectively. It suggests that, in low-energy physics, essentially only the eigenstates of \( L^i_0 \) and \( \bar{L}^i_0 \) corresponding to the eigenvalues 0 and \( \frac{1}{2} \) are important; see also Sect. 7.5.
The analogy

\[
\text{electron, positron} \leftrightarrow \text{heterotic string theory}
\]

\[
\text{positronium} \leftrightarrow \text{type IIA and IIB superstring theory}
\]
suggests that, just as positronium can be realized as a bound state of an electron and a positron, type IIA and IIB superstrings can be realized as “bound states” of heterotic strings. This calls for the study of interacting string theory.

Perturbative string scattering amplitudes can be calculated, in the operator formalism studied in this section, with the help of the Krichever-Novikov generalizations [79] of the Virasoro and super Virasoro algebras for Riemann surfaces of higher genus. But we shall not get into this fairly technical subject. String perturbation theory will not be adequate for the study of non-perturbative phenomena, such as string theory solitons and bound states. Our best bet for getting a first look at such phenomena is the theory of \(D\)-branes, see e.g. the comments in the lectures by R. Dijkgraaf and B. Greene, and in particular [85]. But the problem remains to find a genuinely non-perturbative formulation of interacting string theory; see [80] for some attempts in this direction. Our discussion in Sections 3 and 7.1 indicates where one of the key problems may lie: Space-time \((N, \eta)\) and internal space \((L, G)\) (see eq. (7.29)) will, according to the ideas of Section 3, ultimately be deformed to non-commutative spaces. The study of this deformation calls for a non-perturbative formulation of string theory involving summing over string world-sheets of arbitrary topology, with an arbitrary number of punctures. This sum appears to be ill-behaved [71] — see also Part I of [75] and refs. given there. It is not hard to guess why one runs into problems: Due to the towers of Planck-scale modes of perturbative string theory, whose recoil on space-time geometry is not properly taken into account, the bounds (3.22) and (3.23) on the number of events and the dimension of local “algebras of observables”, respectively, described in Section 3 are violated by perturbative string theory (which, at its outset, treats target- and parameter space as classical). One way out of these difficulties might be to deform the parameter spaces of string world-sheets from classical to non-commutative Riemann surfaces, as envisaged in [76]. One is entitled to expect that it is easier to sum over “all non-commutative Riemann surfaces” than to integrate over the entire moduli space of all classical Riemann surfaces. (But the program alluded to, here, is still in its infancy.)

### 7.3 Some remarks on \(M\)(atrix) models

A proposal inspired by \(M\)-theory [88] currently attracting attention is to trade strings for higher-dimensional extended objects, in particular membranes, with non-commutative parameter spaces. This proposal originates in the thesis of J. Hoppe [81] and in subsequent work of de Wit, Hoppe, Nicolai and others [61], which has recently been reinterpreted and extended by Banks, Fischler, Shenker and Susskind [82]. In this approach, parameter space supersymmetry is replaced by “target space supersymmetry”, see Sect. 5.3.

Let us choose a non-commutative 2-torus as a parameter space (see Section 6). A basis in the “algebra of functions” on the non-commutative 2-torus is given by the \(N \times N\) matrices

\[
T^{(N)}_{\mathcal{L}} := \frac{i}{4\pi} \frac{N}{M} q^{1/2} p_1 p_2 U^{p_1} V^{p_2}, \tag{7.79}
\]
where \( q = e^{4\pi i M/N} \) for two co-prime integers \( M \) and \( N \), \( p_1, p_2 \in \{ -\frac{N-1}{2}, -\frac{N-3}{2}, \ldots , \frac{N-1}{2} \} \) and

\[
U = \begin{pmatrix}
1 & q & 0 \\
q & 1 & 0 \\
0 & q^{N-1} & 1
\end{pmatrix}, \quad V = \begin{pmatrix}
0 & 1 & 0 \\
\cdots & \cdots & 1 \\
1 & 0 & 0
\end{pmatrix};
\]

therefore \( UV = q^{-1}VU \). One checks that

\[
\left[ T^{(N)}_\mathcal{E}, T^{(N)}_\mathcal{Q} \right] = \frac{N}{2\pi M} \sin \left( \frac{2\pi M}{N} (p_1 q_2 - p_2 q_1) \right) T^{(N)}_{\mathcal{E}+\mathcal{Q} \pmod{N}} .
\]

(7.80)

In the limit \( N \to \infty, \frac{M}{N} \to 0 \), these commutation relations approach the relations

\[
\left[ T^{(\infty)}_\mathcal{E}, T^{(\infty)}_\mathcal{Q} \right] = (p_1 q_2 - p_2 q_1) T^{(\infty)}_{\mathcal{E}+\mathcal{Q}}
\]

defining the Lie algebra of functions on the 2-torus with respect to the obvious Poisson bracket. Integration over the non-commutative torus is given by the normalized trace on \( N \times N \) matrices. The \( N \times N \) matrices \( \{ T^{(N)}_\mathcal{E} \} \) span \( gl(N, \mathbb{C}) \). In a unitary representation of the commutation relations (7.80) one has that

\[
\left( T^{(N)}_\mathcal{E} \right)^* = - T^{(N)}_\mathcal{E} \quad \text{and} \quad \text{tr} \left( T^{(N)}_\mathcal{E} \right) = 0
\]

for \( p \neq 0 \). Anti-selfadjoint combinations of these generators span \( su(N) \). In fact, \( su(N) \) is the algebra of “infinitesimal reparametrizations” of the non-commutative 2-torus described by (7.79,80). The “algebra of functions” on this non-commutative torus, \( M_\mathcal{N}(\mathbb{C}) \), is denoted by \( \mathcal{A}^{(N)} \).

In the light-cone gauge (which appears to be incompatible with Poincaré covariance), a (classical) membrane model with parameter space given by the non-commutative 2-torus \( T^2_{(N)} \) with data \( (\mathcal{A}^{(N)}, \mathbb{C}^N, \text{tr} (\cdot)) \), and a target space corresponding to an eleven-dimensional, non-commutative Minkowski space \( \mathbb{M}^{11} \), is described by 9-tuples \( \{ X^1, \ldots , X^9 \} \) of matrices \( X^j \in \mathcal{A}^{(N)}, j = 1, \ldots , 9 \), and a 32-component Majorana spinor \( \Theta \) satisfying \( \Gamma^+ \Theta = 0 \), where \( \Gamma^2 \) is the 32\( \times \)32 Dirac matrix corresponding to the one-form \( d(x^0 - x^{11}) \); the 16 non-zero, independent components \( \theta_\alpha \) of \( \Theta \) are \( N \times N \) matrices of Grassmann generators.

One may view \( \mathbb{M}^{11} \) as a non-commutative space described by a non-abelian “algebra of functions” \( \mathcal{A}^{(\infty)} \) which is an infinite-dimensional \( C^* \)-algebra with a trace \( \text{tr}(\cdot) \). One might expect that an unquantized membrane with parameter space \( T^2_{(N)} \) embedded in \( \mathbb{M}^{11} \) can be described as a \( * \)-homomorphism from \( \mathcal{A}^{(\infty)} \) to \( \mathcal{A}^{(N)} \) (in analogy to the description of a classical sub-manifold embedded in a manifold). However, the right idea appears to be to describe an unquantized membrane as an embedding of \( \mathcal{A}^{(N)} \) into \( \mathcal{A}^{(\infty)} \).

Fixing the center of mass coordinates of an unquantized membrane amounts to imposing the conditions \( \text{tr} (X^j) = \text{tr} (\theta_\alpha) = 0 \) for all \( j = 1, \ldots , 9 \) and \( \alpha = 1, \ldots , 16 \). We may then expand \( X^j \) and \( \theta_\alpha \) in a basis \( \{ T^A \}, A = 1, \ldots , \frac{N(N+1)}{2} - 1 \), of \( su(N) \):

\[
X^j = i X^{jA} T^A, \quad \theta_\alpha = \theta_\alpha^A T^A,
\]

(7.81)

104
where
\[ \text{tr} (T_A T_B) = - \delta_{AB} , \quad [T_A, T_B] = f_{ABC}^C T_C ; \quad (7.82) \]

\( \{ f_{AB} \} \) are structure constants of su(\( N \)), and the summation convention is imposed. The coefficients \( X^{jA} \) are real variables, \( \theta_\alpha^A \) are Grassmann variables, and \( T_A^* = -T_A \) (so that the \( X^{j} \) are Hermitian).

Canonical quantization of the model proceeds as usual: One introduces variables \( P_j^B \) canonically conjugate to \( X^{jA} \) and imposes the commutation relations
\[
\begin{align*}
[X^{jA}, P_k^B] &= i \delta_k^i \delta^{AB} , \quad \{ \theta_\alpha^A, \theta_\beta^B \} = - \delta_{\alpha\beta} \delta^{AB} , \\
[X^{jA}, \theta_\alpha^B] &= [ P_j^A, \theta_\alpha^B ] = 0 .
\end{align*}
\quad (7.83)
\]

These commutation relations have an irreducible *–representation on a Hilbert space \( \mathcal{H}^{(N)} \), which one interprets as the space of state vectors of a quantized membrane or, perhaps more appropriately, of \( N \) \( \theta \)--branes [82,85]. Let \( \{ \gamma^i \}_{i=1}^8 \) be 16x16 symmetric Dirac matrices generating the Clifford algebra \( Cl(\mathbb{R}^8) \), and \( \gamma^9 := \gamma^1 \cdots \gamma^8 \). One defines self-adjoint super-charges on \( \mathcal{H}^{(N)} \) by
\[
D_\alpha := \sum_{\alpha, \beta} \left( P_j^A (\gamma_j^A)^\beta_\alpha + \frac{1}{2} f_{ABC} X^{iB} X^{jC} [\gamma_i, \gamma_j]^\beta_\alpha \right) \theta_\alpha^A .
\quad (7.84)
\]

The Hilbert space \( \mathcal{H}^{(N)} \) carries unitary representations of Spin(9) and of SU(\( N \)): SO(9) is a global symmetry group of the target space \( \mathbb{M}^{11} \), and SU(\( N \)) is the group of reparametrizations of parameter space \( T^2_{(N)} \). The Clifford generators \( \theta_\alpha^A \) and the super-charges \( D_\alpha \) transform as spinors under the adjoint action of the representation of Spin(9) on \( \mathcal{H}^{(N)} \); the generators \( X^{jA}, P_j^A \) transform as vectors under Spin(9) and in the adjoint representation under SU(\( N \)); the generators \( \theta_\alpha^A \) transform in the adjoint representation of SU(\( N \)), and the super-charges \( D_\alpha \) are SU(\( N \))–invariant.

Next one computes the anti-commutators \( \{ D_\alpha, D_\beta \} \) and finds that
\[
\{ D_\alpha, D_\beta \} = 2 \delta_{\alpha\beta} H^{(N)} + 2 X^{jA} (\gamma_j)^{\alpha\beta} L_A ,
\quad (7.85)
\]
where \( H^{(N)} \) is the light-cone Hamiltonian, i.e., (classically) the generator of translations along the light rays \( x^0 + x^{10} = \text{const} \), and \( L_A = i L(T_A) \) are self-adjoint generators of the unitary representation of SU(\( N \)) on \( \mathcal{H}^{(N)} \) — compare to the structure described in eq. (5.137). The space of physical state vectors of the theory is the subspace \( \mathcal{H}_0^{(N)} \) of reparametrization-invariant, i.e., SU(\( N \))–invariant vectors in \( \mathcal{H}^{(N)} \). On this subspace, the relations (7.85) reduce to
\[
\{ D_\alpha, D_\beta \} = 2 \delta_{\alpha\beta} H^{(N)}
\quad (7.86)
\]
so that \( \left( \mathcal{A}_0, \mathcal{H}_0^{(N)}, \{ D_\alpha \}_{a=1}^{16} \right) \) are \( N = 16 \) supersymmetric spectral data, where \( \mathcal{A}_0 \) is the largest *–subalgebra of \( B(\mathcal{H}_0^{(N)}) \) with the property that for every \( a \in \mathcal{A}_0, [D_\alpha, a] \) is a bounded operator on \( \mathcal{H}_0^{(N)} \).

The light-cone Hamiltonian \( H_0^{(N)} := H^{(N)} \mid_{\mathcal{H}_0^{(N)}} \) is clearly a positive, self-adjoint operator on \( \mathcal{H}_0^{(N)} \). Its spectrum covers the half-axis \([0, \infty)\); see [61]. One expects that the spectrum of \( H_0^{(N)} \) is purely absolutely continuous, except for a possible finitely-degenerate eigenvalue at 0; see [83] for some preliminary results.
The model discussed here fits nicely into the general framework considered in Sect. 5.3. It is quite clear that “physically relevant” results can only be expected to emerge in a limiting regime as $N \to \infty$ (with $\frac{M}{N}$ in (7.79,80) approaching 0 or an irrational number). Lots of conjectures about such limiting regimes have recently been discussed; see e.g. [82,84].

Attempts to interpret these models as a formulation of some sort of non-perturbative quantum gravity appear slightly premature: Global symmetries of target space-time should not enter a formulation of quantum gravity, and the “light-cone gauge” is not a meaningful concept, in general. In this respect, perturbative string theory is at a much more advanced stage. Yet, some of the problems arising in the analysis of the matrix models considered above are, of course, interesting, at least mathematically.

A generalization of these matrix models in the form of dimensionally reduced super Yang-Mills theories appears in the study of $D$–branes in superstring theory [85]. The gauge group is $\text{U}(N)$, where $N$ is the number of $D$–branes. An action functional for $N$ parallel $D$–branes of dimension $p < 10$ can be obtained using the Connes-Lott construction [50]. One starts from the algebra

$$\mathcal{A}^{(N,p)} := C^\infty (M) \otimes M_N (\mathbb{C}),$$

where $M$ is a $(p+1)$–dimensional manifold parametrizing the world-volume of a $D$–brane, and considers $N = 1$ supersymmetric spectral data

$$(\mathcal{A}^{(N,p)}, \mathcal{H}^{(N,p)}, D^{(N,p)}) ,$$

(7.87)

where $\mathcal{H}^{(N,p)}$ is the Hilbert space of square-integrable spinors on $M$ with values in $M_N (\mathbb{C})$, and the Dirac operator is given by

$$D^{(N,p)} = D_M + \sum_{j=p+1}^{9} \gamma^j X_j \quad \text{with} \quad D_M = \sum_{\mu=0}^{p} \gamma^\mu \nabla_\mu ;$$

(7.88)

$\nabla$ is the Levi-Civita connection on $M$, the matrices $\gamma^0, \ldots, \gamma^9$ are $32 \times 32$ Dirac matrices, and $X_{p+1}, \ldots, X_9$ are commuting $N \times N$ matrices describing the transversal coordinates of $N$ $D$–branes in the ground state configuration. The form (7.88) of the Dirac operator is derived from (open) superstring theory [85]. A (low-energy effective) action functional for $N$ parallel, $p$–dimensional $D$–branes can be obtained from (7.87,88) e.g. by following the constructions in [50,89].

Of course, it is presumably not correct to describe the world-volumes of $D$–branes as classical manifolds. Our proposal is to replace them by non-commutative spaces described by spectral data $(\mathcal{B}_p \otimes M_N (\mathbb{C}), \mathcal{H}, D)$, where $\mathcal{B}_p$ is a non-abelian “algebra of functions” on the world-volume. From these data one can construct Yang-Mills(-Higgs) action functionals as in [5,50]. If $\mathcal{B}_p$ is a finite-dimensional matrix algebra it is not difficult to quantize the systems described by these action functionals using functional integrals. Reasons why the world-volumes of $D$–branes might typically be non-commutative spaces will become apparent in the next section.
7.4 Two-dimensional conformal field theories

In Sect. 7.2, we have observed that unitary (super-)conformal field theories play a fundamental role in the study of string theory vacua when one is able to “separate variables”, see (7.54–58,78). They ought to describe the geometry of “internal spaces”, denoted \((L, G)\) in (7.29). This idea has motivated a program initiated in [78,24], and stimulated, in part, by the work in [15,16,5]: to reconstruct loop space and target space geometries from algebraic data provided by super-conformal field theories. The observation is that, in general, those geometries are non-commutative geometries, in the sense of Connes [5]. To develop this theme would require more room than is left. We refer the reader to [24,78,90,107] for various technical aspects of this program, but hasten to add that much more technical work remains to be done.

1) Recap of two-dimensional, local quantum field theory

Parameter space-time is chosen to be a two-dimensional cylinder \(\Sigma\) with coordinates \((\sigma, \tau)\), \(0 \leq \sigma < 2\pi, \tau \in \mathbb{R}\), equipped with a Lorentz metric \(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\). We consider a local, relativistic quantum field theory on \(\Sigma\), see [91], with a Hilbert space \(H\) of physical state vectors carrying a continuous, unitary representation of the group of translations on \(\Sigma\) with infinitesimal generators \(H (\tau\text{-translations})\) and \(P (\sigma\text{-translations})\) such that \(H \pm P \geq 0\) (7.89) (spectrum condition). It is also assumed that \(H\) contains a vector \(\Omega\), the “vacuum vector”, with the property that \((H \pm P)\Omega = 0\). The vacuum vector is assumed to be a cyclic vector for a \(*\)-algebra generated by local field operators \(\{\varphi_I(\xi)\}_{I \in J}, \xi = (\sigma, \tau) \in \Sigma\). Local bosonic field operators satisfy locality in the form

\[
\left[\varphi_I(\xi), \varphi_J(\eta)\right] = 0,
\]

(7.90) for all \(I, J\) in \(J\), whenever \((\xi - \eta)^2 < 0\) (i.e., whenever \(\xi\) and \(\eta\) are space-like separated). The fields \(\varphi_I(\xi)\) are operator-valued tempered distributions with the usual properties described in [91]. The fields \(\{\varphi_I(f) \mid I \in J, f \in \mathcal{S}(\Sigma), \text{supp } f \subset O\}\) where \(O\) is a contractible open region in \(\Sigma\) (specifically a contractible “diamond”), form a \(*\)-algebra \(\mathcal{A}(O)\) of unbounded operators defined on an invariant domain \(D\) dense in \(H\). By (7.89) and (7.90), the vacuum \(\Omega\) is a cyclic and separating vector for \(\mathcal{A}(O)\), [91].

We assume that, among the local bosonic fields \(\varphi_I(\xi)\) of the theory, there is a field \(T_{\mu\nu}(\xi)\), the energy-momentum tensor of the theory, such that

\[
H = \int_{\tau = \text{const.}} T_{00}(\sigma, \tau) \, d\sigma, \quad P = \int_{\tau = \text{const.}} T_{01}(\sigma, \tau) \, d\sigma.
\]

(7.91)

Wightman’s reconstruction theorem [91] asserts that the entire structure of a local relativistic quantum field theory is encoded in its Wightman distributions

\[
W_{I_1...I_n}(\xi_1, \ldots, \xi_n) := \langle \Omega, \prod_{j=1}^{N} \varphi_{I_j}(\xi_j) \Omega \rangle.
\]

(7.92)

By (7.89), these distributions are boundary values of functions \(W_{I_1...I_n}(\xi_1, \ldots, \xi_n)\) analytic in \(\zeta_1, \ldots, \zeta_n\) on the domain

\[
\left\{ (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^{2n} \middle| \text{Im} (\zeta_{j+1} - \zeta_j) \in V_+ \right\}.
\]
where $V_+ = \{ (\sigma, \tau) \mid \tau > |\sigma| \}$ is the forward light cone, and by (7.90) the domain of analyticity can be extended to the “permuted forward tube”, \( \text{Im}(\zeta_{\pi(j+1)} - \zeta_{\pi(j)}) \in V_+, \pi \in S_n \), which contains the Euclidean region

\[
\{ (\zeta_1, \ldots, \zeta_n) \mid \zeta_j = (\sigma_j, i\tau_j), (\sigma_j, \tau_j) \in \mathbb{R}^2 \}
\]

(7.93)

One defines the Schwinger functions by

\[
S_{I_1 \ldots I_n} (\xi_1, \ldots, \xi_n) := W_{I_1 \ldots I_n} ((\sigma_1, i\tau_1), \ldots, (\sigma_n, i\tau_n))
\]

(7.94)

The key result concerning Schwinger functions is the Osterwalder-Schrader reconstruction theorem [92]. Defining 

\[
\phi_{I} (\sigma, \tau) := e^{-\tau H} \phi_{I}(\sigma, 0) e^{\tau H}
\]

(7.95)

and

\[
\psi := \prod_{j=1}^{n} \phi_{I_j} (\sigma_j, \tau_j) \Omega, \quad 0 < \tau_1 < \tau_2 < \ldots < \tau_n,
\]

(7.96)

with $I_1, \ldots, I_n \in J$ and $n = 0, 1, 2, \ldots$. The scalar products \( \langle \psi, \psi' \rangle \), with $\psi$ and $\psi'$ as in (7.95) can then be expressed in terms of the Schwinger functions introduced in (7.94); see [92].

It is sometimes convenient (“radial quantization”) to introduce the variables

\[
z = e^{-\tau + i\sigma}, \quad \bar{z} = e^{-\tau - i\sigma},
\]

(7.96)

with $(\sigma, \tau) \in \mathbb{C}^2$. The Euclidean region (7.93) corresponds to $\bar{z}_j = z_j^*$ (\( \equiv \) complex conjugate of $z_j$) for $j = 1, \ldots, n$.

2) Conformal field theory [99]

A relativistic quantum field theory is \textbf{M"obius-invariant} if there are positive numbers (conformal weights) $h_I, \bar{h}_I, I \in J$, such that the forms

\[
W_{I_1 \ldots I_n} (z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n) \prod_{j=1}^{n} (d z_j)^{h_{I_j}} (d \bar{z}_j)^{\bar{h}_{I_j}}
\]

(7.97)

are invariant under Möbius transformations

\[
z_j \mapsto \frac{az_j + b}{cz_j + d}, \quad \bar{z}_j \mapsto \frac{a^* \bar{z}_j + b^*}{c^* \bar{z}_j + d^*}, \quad j = 1, \ldots, n,
\]

(7.98)

for \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C}) \), for arbitrary $I_1, \ldots, I_n$ in $J$, and for all $n$, and if the generators of the virtual representation [100] of the Möbius group on $\mathcal{H}$ can be expressed in terms of Fourier modes

\[
\frac{1}{2\pi} \int_{0}^{2\pi} T_{\mu\nu}(\sigma, 0) e^{in\sigma} \, d\sigma, \quad n = 0, \pm 1,
\]

of the energy-momentum tensor.

A theorem due to Lüscher and Mack [93] says that if a local, relativistic quantum field theory on $\Sigma$ is Möbius-invariant in the sense just described, then it is a \textbf{conformal field theory}, i.e. $(u_\pm := \tau \pm \sigma)$

\[
T_{\mu\nu}(\xi) = \begin{pmatrix} 0 & T_{++}(u_+) \\ T_{--}(u_-) & 0 \end{pmatrix},
\]

(7.99)
with \( T(\xi) \equiv \text{tr} \left[ T_{\mu\nu}(\xi) \right] = 2T_{++}(\xi) = 0 \) and

\[
\frac{\partial}{\partial u_-} T_{++} = \frac{\partial}{\partial u_+} T_{--} = 0 ,
\]

and the Fourier modes

\[
L_n = \frac{1}{2\pi} \int_0^{2\pi} T_{++}(u_+) \, e^{inu_+} \, du_+ ,
\]

\[
\bar{L}_n = \frac{1}{2\pi} \int_0^{2\pi} T_{--}(u_-) \, e^{inu_-} \, du_- ,
\]

(7.100)

\( n \in \mathbb{Z} \), span two commuting Virasoro algebras, \( \text{Vir} \) and \( \overline{\text{Vir}} \), with structure relations as in eq. (7.36). The energy-momentum tensor is a conformal tensor of dimension 2. Recalling that \( z = e^{-\tau + i\sigma}, \bar{z} = e^{-\tau - i\sigma} \) in the Euclidean region \( \{ \xi = (\sigma, i\tau) \mid 0 \leq \sigma < 2\pi, \tau > 0 \} \), this motivates us to define

\[
T(z) := z^{-2} T_{++}(\tau + i\sigma) , \quad \overline{T}(\bar{z}) := \bar{z}^{-2} T_{--}(\tau - i\sigma) .
\]

(7.101)

Then

\[
L_n = \frac{1}{2\pi i} \oint_{|z|=1} z^{n+1} T(z) \, dz ,
\]

\[
\bar{L}_n = \frac{1}{2\pi i} \oint_{|\bar{z}|=1} \bar{z}^{n+1} \overline{T}(\bar{z}) \, d\bar{z} .
\]

(7.102)

Under somewhat stronger hypotheses, one can prove that, in a conformal field theory, the domain of analyticity of the Wightman functions \( W_{I_1...I_n}(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n) \) is given by

\[
M_n \times \overline{M}_n ,
\]

(7.103)

where \( M_n \) is the universal covering of \( \{ z_1, \ldots, z_n \mid z_i \neq z_j \text{ for } i \neq j \} \), and analogously for \( \overline{M}_n \); see [97].

In conformal field theory, one should attempt to find all local fields \( \psi^{(K)}(u_\pm) , K \in \mathcal{K}_\pm \) for some index sets \( \mathcal{K}_\pm \), with the property that \( \frac{\partial}{\partial u_\pm} \psi^{(K)}(u_\pm) = 0 \), i.e., all local chiral fields (depending only on one of the two light-cone coordinates). Obviously, \( T_{++} \) and \( T_{--} \) are examples of local chiral fields of conformal dimension 2. Local chiral fields are completely determined by their Fourier modes

\[
\psi_n^{(K)} = \frac{1}{2\pi} \int_0^{2\pi} \psi^{(K)}(u_+) \, e^{inu_+} \, du_+ ,
\]

(7.104)

and, if \( \psi^{(K)} \) is a conformal tensor of weight \( (h_K, 0) \) independent of \( u_- \) we find that

\[
\psi_n^{(K)} = \frac{1}{2\pi i} \oint_{|z|=1} z^{n+h_K-1} \psi^{(K)}(z) \, dz .
\]

(7.105)
A similar equation holds for the Fourier modes \( \bar{\psi}_n^{(K)} \) of a chiral field \( \psi^{(K)}(u_-), K \in \mathcal{K}_- \), which is assumed to be a conformal tensor of weight \((0,\bar{h}_K)\). These definitions make sense provided \( h_K \in \mathbb{N} \), i.e., for local, chiral Bose fields. They can be extended to local, chiral Fermi fields \( \psi^{(K)}(z) \) with \( h_K + \frac{1}{2} \in \mathbb{N} \), after one has chosen a spin structure on the circle \( 0 \leq \sigma < 2\pi \), i.e., either periodic (“Ramond”) or anti-periodic (“Neveu-Schwarz”) boundary conditions. For periodic boundary conditions, the Fourier modes are labeled by integers; for anti-periodic boundary conditions, they are labeled by half-integers.

A chiral field \( \psi^{(K)} \) is a Bose field iff
\[
\left[ \psi^{(K)}(u_+), \psi^{(K')}(u'_+) \right] = 0 \tag{7.106}
\]
for \( u_+ \neq u'_+ \), where \( \psi^{(K)} \) is an arbitrary chiral Bose- or Fermi field; then \( h_K \in \mathbb{N} \). Chiral fields \( \psi^{(K)} \) and \( \psi^{(K')} \) are Fermi fields iff
\[
\left\{ \psi^{(K)}(u_+), \psi^{(K')}(u'_+) \right\} = 0 \tag{7.107}
\]
for \( u_+ \neq u'_+ \); then \( h_K + \frac{1}{2}, h_K' + \frac{1}{2} \in \mathbb{N} \).

We shall always assume that there is an involution \( ^+ : \mathcal{K}_+ \rightarrow \mathcal{K}_+ \) such that
\[
(\psi^{(K)}_n)^* = \psi^{(K^+)} \tag{7.108}
\]
or, equivalently, \( \psi^{(K)}(u_+)^* = \psi^{(K^+)}(u_+) \).

The chiral algebra \( \mathcal{E} \) of a conformal field theory on the cylinder \( \Sigma \) is the unital \( ^* \)-algebra of (generally unbounded) operators on \( \mathcal{H} \) generated by \( 1 \) and \( \left\{ \psi^{(K)}_n \mid n \in \mathbb{Z}, K \in \mathcal{K}_+, h_K \in \mathbb{N} \right\} \), with \( \psi^{(K)}_n \) as in (7.104,105). The anti-chiral algebra \( \bar{\mathcal{E}} \) is defined similarly. The conformal field theory is left–right symmetric iff
\[
\mathcal{E} \cong \bar{\mathcal{E}}, \tag{7.109}
\]
i.e., iff \( \mathcal{E} \) and \( \bar{\mathcal{E}} \) are \(^*\)-isomorphic \(^*\)-algebras.

One can define \( \mathbb{Z}_2 \)-graded, extended (anti-)chiral algebras \( \mathcal{C}_\alpha, \bar{\mathcal{C}}_\alpha \) with \( \alpha \) = Ramond or \( \alpha \) = Neveu-Schwarz, by including in their definition the Fourier modes of chiral Fermi fields with periodic or anti-periodic boundary conditions, respectively. Then \( \mathcal{C}_\# \) is the even part of \( \mathcal{C}_\alpha\#. \) The algebra \( \mathcal{C}_\alpha \) is the universal enveloping algebra of a graded Lie algebra iff its generators obey relations
\[
\left[ \psi^{(K)}_n, \psi^{(K')}_m \right]_g = f^{KK'}_{K''}(n,m) \psi^{(K'')}_{n+m} + g^{KK'}(n) \delta_{n+m,0} \tag{7.110}
\]
for some structure constants \( f^{KK'}_{K''}(n,m) \) and “central elements” \( g^{KK'}(n) \).

By applying the calculus of residues to local chiral fields one finds that a chiral algebra \( \mathcal{E} \) can be equipped with a family \( \{ \Delta_z \mid z \in \mathbb{C}\}^* \) of co-products \( \Delta_z : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{E} \), defined by
\[
\Delta_z \left( \psi^{(K)}_n \right) = \delta_z \left( \psi^{(K)}_n \right) \otimes 1 + 1 \otimes \psi^{(K)}_n ,
\]
where
\[
\delta_z \left( \psi^{(K)}_n \right) := \sum_{m=0}^{\infty} \binom{n+h_K-1}{m} z^{n-m+h_K-1} \psi^{(K)}_{m+h_K+1} \quad \text{for } n > -h_K ,
\]
and
\[
\delta_z \left( \psi^{(K)}_n \right) := \sum_{m=0}^{\infty} (-1)^m \left( \binom{m-n-h_K}{m} z^{m+n+h_K-1} \psi^{(K)}_{m+h_K+1} \right) \quad \text{for } n \leq -h_K .
\]
see e.g. [96]. These co-products can obviously be extended to maps

\[ \Delta_z : \mathcal{C}_{NS} \longrightarrow \mathcal{C}_{NS} \otimes \mathcal{C}_{NS} \]
\[ C_R \longmapsto \mathcal{C}_{NS} \otimes C_R . \]  

(7.112)

Clearly \( \delta_z(1) = 0 \), so that \( \Delta_z(1) = 1 \otimes 1 \).

Conformal field theory can be viewed as the representation theory of a pair of a chiral algebra \( \mathcal{E} \) and an anti-chiral algebra \( \mathcal{E}^\# \) which are always assumed to contain \( U(Vir) \), \( U(Vir) \), respectively — i.e., \( T^\#(z^\#) \) is among the generators of \( \mathcal{E}^\# \). A unitary \(^*\)–representation \( j \) of \( \mathcal{E}^\# \) is a \(^*\)–homomorphism from \( \mathcal{E}^\# \) to a \(^*\)–algebra of densely defined, unbounded operators on a Hilbert space \( \mathcal{H}_j \) such that

\[ j(\psi_n^K) = j\left(\psi^\#_{-n}^{(K)}\right) . \]  

(7.113)

Because of the relativistic spectrum condition (7.89) we only consider “positive-energy representations”: These are representations \( j \) of \( \mathcal{E}^\# \) with the property that \( j(L^\#_0) \) is a positive self-adjoint operator

\[ j(L^\#_0) \geq 0 . \]  

(7.114)

Note that, formally, \( j(L^\#_0) = \frac{1}{2}(H \pm P)|_{\mathfrak{h}_j} \), which according to (7.89) must be positive operators.

It follows from (7.102) and (7.105) that

\[ \left[ L^\#_0, \psi_n^K \right] = -n\psi_n^K, \quad n \in \mathbb{Z} (+1/2) , \]  

(7.115)

hence \( \mathcal{E} \) and \( \mathcal{E}^\# \) are \( \mathbb{Z} \)–graded algebras. Suppose that \( \chi \) is a vector in \( \mathfrak{h}_j \) with the property that

\[ \langle \chi, j(L^\#_0) \chi \rangle \leq \left( h^\#_j + \varepsilon \right) \langle \chi, \chi \rangle , \]  

(7.116)

where \( h^\#_j := \inf \text{spec } j(L^\#_0) \) and \( \varepsilon < \frac{1}{2} \). Combining (7.115) and (7.116), we conclude that

\[ j\left(\psi_n^K\right) \chi = 0 \quad \text{for all } n > 0 . \]  

(7.117)

A vector \( \chi \) satisfying (7.117) is called a “highest weight vector”. If the representation space \( \mathfrak{h}_j \) is separable it follows from (7.114–117) that it can be decomposed into a direct integral of unitary “highest weight” modules for \( \mathcal{E}^\# \). A vacuum representation \( \varepsilon \) of an (anti-)chiral algebra \( \mathcal{E}^\# \) is an irreducible, unitary positive-energy representation of \( \mathcal{E}^\# \) on a Hilbert space \( \mathcal{H}_\varepsilon^\# \) containing a highest weight vector \( \Omega \) (the vacuum), i.e.,

\[ \varepsilon(\psi_n^K) \Omega = 0 \quad \text{for all } n > 0 \]  

(7.118)

for all \( K \in \mathcal{K}_\pm \) which moreover is Möbius-invariant, i.e.,

\[ \varepsilon(L_{\pm 1}) \Omega = \varepsilon(L_0) \Omega = 0 . \]  

(7.119)

In the following, we usually omit the symbol \( \varepsilon \).

A chiral algebra \( \mathcal{E}^\# \) is called rational iff it only has a finite number of irreducible unitary positive-energy representations \( j \), including a unique vacuum representation \( \varepsilon \).
Examples are the universal enveloping algebras of the Virasoro algebras with central charge \( c = 1 - \frac{1}{p(p+1)} \), \( p = 1, 2, \ldots \), and of simply-laced Kac-Moody algebras at integer level.

Thanks to the existence of the co-products \( \Delta_z \) defined in (7.111), the irreducible unitary positive-energy representations of a rational (anti-)chiral algebra \( E^\# \) form the irreducible objects of a semi-simple, rigid, braided \( C^* \)-tensor category \( \mathcal{T} \), with sub-objects of direct sums (see e.g. [98]): The vacuum representation \( e \) plays the role of the unit in \( \mathcal{T} \), as one derives without difficulty from (7.111). Given two unitary positive-energy representations \( j \) and \( k \) of \( E^\# \), one defines their tensor product

\[
j \otimes_z k := (j \otimes k) \circ \Delta_z ,
\]

which is independent of \( z \) up to isomorphism. Then we have

\[
e \otimes_z j \cong j \otimes_z e \cong j .
\]

Moreover, given an irreducible unitary positive-energy representation \( j \), one can show (in the proper setting: see [101,102,103,12]) that there exists a unique irreducible unitary positive-energy representation \( j^\vee \), the representation conjugate to \( j \), such that \( j \otimes_z j^\vee \cong j^\vee \otimes_z j \) contains \( e \) as a sub-representation precisely once.

Let \( I \) denote the finite set of all irreducible, unitary positive-energy representation of a rational chiral algebra \( E \). The fusion rule algebra is the abelian algebra generated by \( \{ N_i^j \} \), where \( N_i^j \) is the multiplicity of \( i \in I \) in the tensor product representation \( j \otimes_k k \cong k \otimes_z j \), for \( j \) and \( k \) in \( I \). Then

\[
\sum_l N_i^l N_j^l = \sum_l N_i^l N_k^l , \quad N_i^j = N_i^j \vee , N_i^j \vee = N_i^j \vee .
\]

Given three representations \( i, j \) and \( k \) in \( I \), there exist intertwiners

\[
V_i^\alpha (\chi_j, z)_k : \mathfrak{h}_k \to \mathfrak{h}_i ,
\]

\( \chi_j \in \mathfrak{h}_j \), \( z \in \mathbb{C} \), \( \alpha = 1, \ldots, N_i^j \), such that

\[
i (\psi_n^{(K)}) V_i^\alpha (\chi_j, z)_k - V_i^\alpha (\chi_j, z)_k k (\psi_n^{(K)}) = V_i^\alpha (\delta_z (\psi_n^{(K)})_n \chi_j, z)_k ,
\]

for every generator \( \psi_n^{(K)} \) of \( E \), with \( \delta_z \) as in (7.111).

The intertwiners \( V_i^\alpha (\chi_j, z)_k \) are called chiral vertex operators and obey braid commutation relations and fusion equations involving braiding matrices \( R^\pm [i,j,l,m]_{k\gamma\delta} \) which are solutions of the celebrated polynomial equations, see [94,95,96].

In “radial quantization”, the product

\[
V_i^\alpha (\chi_j, z_2)_k V_k^\beta (\chi_1, z_1)_m
\]

is a well-defined operator from \( \mathfrak{h}_m \) to \( \mathfrak{h}_i \) provided \( |z_1| < |z_2| \). Matrix elements of this product have an analytic continuation \( (z_1, z_2) \) along the paths \( \gamma^+ \) and \( \gamma^- \), with
\[ \frac{\gamma^+}{\gamma^-} \leftrightarrow z_2 \leftrightarrow z_1, \quad \frac{\gamma^-}{\gamma^+} \leftrightarrow z_2 \leftrightarrow z_1, \]

and
\[
\left[ V_i^\alpha(\chi_j, z_1)_k V_k^\beta(\chi_l, z_2)_m \right]_{\gamma^\pm} = \sum R^\pm [i, j, l, m]_{k, j, \alpha, \beta} V_i^n(\chi_j, z_2)_n V_k^n(\chi_l, z_1)_m. \tag{7.125}
\]

Furthermore,
\[
V_i^\alpha(\chi_j, z_1)_k V_k^\beta(\chi_l, z_2)_m = \sum F[i, j, l, m]_{n, \alpha, \beta} V_i^n(\chi_j, z_2)_l V_k^n(\chi_l, z_2)_m, \tag{7.126}
\]

where \( F[i, j, l, m]_{n, \alpha, \beta} \) are the fusing matrices; see [94,95,96].

A chiral vertex operator \( V_i^\alpha(\chi_j, z)_k \) is called primary iff \( \chi_j \in h_j \) is a highest weight vector for \( E \). It is not hard to show [99,97] that a primary chiral vertex operator satisfies the differential equations
\[
i(L_n) V_i^\alpha(\chi_j, z)_k - V_i^\alpha(\chi_j, z)_k k(L_n) = \left[ z^{n+1} \frac{d}{dz} + z^n (n+1) h_j \right] V_i^\alpha(\chi_j, z)_k, \tag{7.127}
\]

for all \( n \in \mathbb{Z} \), where \( h_j \) is the eigenvalue of \( j(L_0) \) (the “highest weight”) corresponding to the eigenvector \( \chi_j \).

From the chiral vertex operators of a chiral algebra \( E \) and an anti-chiral algebra \( \bar{E} \) one can attempt to construct local fields \( \varphi^\alpha_{\chi_j \otimes \chi_j'}(z, \bar{z}) \) by setting
\[
\varphi^\alpha_{\chi_j \otimes \chi_j'}(z, \bar{z}) := \sum D[i, j, l, m]_{k, j, \alpha, \beta} V_i^\gamma(\chi_j, z)_l V_i^\gamma(\chi_j, \bar{z})_m; \tag{7.128}
\]

the \( D \)’s are complex “sewing coefficients”. Here \( V_i^\alpha(\ldots)_{k'} |_{h_{i'}} = 0 \) if \( k' \neq l' \). These local fields are operator-valued distributions from the Hilbert space
\[
\mathcal{H} = \bigoplus_{(k, \bar{k}) \in \Pi} h_k \otimes h_{\bar{k}} \otimes \mathbb{C}^{n(k, \bar{k})} \tag{7.129}
\]
to itself, where \( \Pi \) is a subset of the product \( I \times \bar{I} \) of irreducible unitary positive-energy representations of \( E \) and \( \bar{E} \), determined by the set of non-zero sewing coefficients \( D \), and \( \mathbb{C}^{n(k, \bar{k})} \) is a “multiplicity space” corresponding to the index \( \alpha = 1, \ldots, n(k, \bar{k}) \) which labels different left-right sewings.

Locality is the constraint that \( \varphi^\alpha_{\chi_j \otimes \chi_j'} \) and \( \varphi^\beta_{\chi_k \otimes \chi_k} \) commute whenever their arguments, \((\sigma, \tau)\) and \((\sigma', \tau')\), are space-like separated. This constraint yields over-determined algebraic equations for the sewing coefficients \( D \) in terms of matrix elements of the braid
matrices $R^\pm$, see e.g. [97]. Examples of solutions of these equations can be found in [104,105] (and in the refs. given there). Note that, by (7.124),

$$\left[ \psi_n^{(K)}, \varphi_{\chi_j \otimes \chi_j}^\omega (z, \bar{z}) \right] = \varphi_{\delta \omega (\psi^{(K)})}^\omega \chi_j \otimes \chi_j (z, \bar{z}) . \quad (7.130)$$

This equation can be understood by applying eq. (7.105) and Cauchy’s theorem to the l.s., and this calculation has originally motivated the definition of the co-products in (7.111).

From eqs. (7.125,126) and (7.128) one can derive the so-called operator product expansion (OPE) of two local conformal fields, see [99]: Let $\chi_{l \#} \in \mathfrak{h}_{\#}$ be an eigenvector of $i^\# (L_0^\#)$ corresponding to an eigenvalue $h_{i \#} \geq 0$ for $i^\# = j^\#, k^\#, l^\#$. There are invariant tensors $C (\chi_j, \chi_j, \alpha \mid \chi_k, \chi_k, \beta \mid \chi_l, \chi_l, \gamma)$ such that

$$\varphi_{\chi_j \otimes \chi_j}^\alpha (z, \bar{z}) \varphi_{\chi_k \otimes \chi_k}^\beta (w, \bar{w}) = \sum C (\chi_j, \chi_j, \alpha \mid \chi_k, \chi_k, \beta \mid \chi_l, \chi_l, \gamma) (z - w)^{-h_j - h_k + h_l} \times (\bar{z} - \bar{w})^{-h_j - h_k + h_l} \varphi_{\chi_l \otimes \chi_l}^\gamma (w, \bar{w}) , \quad (7.131)$$

where the sum extends over a complete, orthonormal set of vectors $\chi_{l \#} \in \mathfrak{h}_{\#}$, over all $l$ and $\bar{l}$ and all $\gamma$. The coefficients $C$ on the r.s. of (7.131) can be expressed in terms of the fusing matrices $F$, the sewing coefficients $D$ and matrix elements of chiral vertex operators.

Local commutativity of $\varphi_{\chi_j \otimes \chi_j}^\gamma (w, \bar{w})$ implies that $h_l - h_\bar{l} \in \mathbb{Z}$. It follows from general results of local, relativistic quantum field theory that the operators

$$\varphi_{\chi_j \otimes \chi_j}^\alpha (\sigma, \bar{\tau}) := \int_{-\infty}^{\infty} d\tau f(\tau) \varphi_{\chi_j \otimes \chi_j}^\alpha (\sigma, \tau) , \quad (7.132)$$

where $(\sigma, \tau) \in \Sigma$ and $f$ is an arbitrary Schwartz space test function, are densely defined operators on $\mathcal{H}$. They generate a unital algebra $\mathcal{F}$ of “functions on quantized phase space over loop space” — compare to eq. (2.9).

The representation-theoretic approach to local conformal field theory outlined in this subsection (see [99,94,95,96,97] and refs. given there) can be translated into the general framework of algebraic quantum field theory [101,102,103,12,98], where one works with $^*$-algebras and von Neumann algebras of bounded operators. This offers considerable advantages in rendering the analysis mathematically rigorous but makes the theory more abstract. There is no room to review the algebraic approach in these notes.

3) A dictionary between conformal field theory and Lie group theory

We consider a compact, semi-simple Lie group $G$ with Lie algebra $\mathfrak{g}$ as in Sect. 4.2. Let $\mathcal{R} = \mathcal{R}_G$ denote the list of all irreducible representations of $G$, and let $\mathcal{H}_G = L^2(G, dg)$, where $dg$ is the Haar measure on $G$. By the Peter-Weyl theorem,

$$\mathcal{H}_G = \bigoplus_{l \in \mathcal{R}_G} W_l \otimes W_{l^\vee} , \quad (7.133)$$

$^*$“Smearing” $\varphi_{\chi_j \otimes \chi_j}^\alpha (\sigma, \tau)$ in $\sigma$, for fixed $\tau$, does usually not yield a well-defined operator!
where $W_I$ is the representation space for the representation $I$; see (4.79). Let $\mathfrak{g}_L$ (resp. $\mathfrak{g}_R$) denote the Lie algebra of left (resp. right) invariant vector fields on $G$.

Comparing the summary of group representation theory presented in Sect. 4.2 with the review of two-dimensional conformal field theory in the last subsection, we arrive at the following dictionary.

<table>
<thead>
<tr>
<th>Lie group theory</th>
<th>Conformal field theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>compact, semi-simple</td>
<td>two-dimensional (rational)</td>
</tr>
<tr>
<td>Lie group $G$</td>
<td>conformal field theory $\mathcal{Q}$</td>
</tr>
<tr>
<td>$\mathcal{H}_G$, eq. (7.133)</td>
<td>$\mathcal{H}$, eq. (7.129)</td>
</tr>
<tr>
<td>$U(\mathfrak{g}_L)$</td>
<td>$\mathcal{E}$</td>
</tr>
<tr>
<td>$U(\mathfrak{g}_R)$</td>
<td>$\mathcal{E}$</td>
</tr>
<tr>
<td>$\mathcal{R}_L \cong \mathcal{R}$</td>
<td>$I$</td>
</tr>
<tr>
<td>$\mathcal{R}_R \cong \mathcal{R}^\vee \cong \mathcal{R}$</td>
<td>$\bar{I}$</td>
</tr>
<tr>
<td>$\triangle : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$</td>
<td>$\triangle_z$ as in (7.111)</td>
</tr>
<tr>
<td>$g \ni X \mapsto X \otimes 1 + 1 \otimes X$</td>
<td>$N^k_{ij}$ as in (7.122)</td>
</tr>
<tr>
<td>$N^k_{ij}$ as in (4.80)</td>
<td>$V^\alpha_i(\chi_j, z)_k$, eq. (7.123)</td>
</tr>
<tr>
<td>$V^\alpha(I, J</td>
<td>K)$, eq. (4.81)</td>
</tr>
<tr>
<td>$C(I, J</td>
<td>k)$, eq. (4.82)</td>
</tr>
<tr>
<td>$C(G)$ (algebra of continuous functions on $G$)</td>
<td>$H = L_0 + \bar{L}_0$ (Hamilton operator on $\mathcal{H}$)</td>
</tr>
<tr>
<td>Laplace–Beltrami operator $\triangle_G$ on $\mathcal{H}_G$</td>
<td>algebra $\mathcal{F}$ defined below eq. (7.132)</td>
</tr>
<tr>
<td>algebra $\mathcal{F}_\hbar$ of “functions on quantized phase space” over $G$, see eq. (2.9)</td>
<td>spectral data $(\mathcal{F}, \mathcal{H}, H = L_0 + \bar{L}_0)$</td>
</tr>
<tr>
<td>spectral data $(\mathcal{F}_\hbar, \mathcal{H}_G, \triangle_G)$</td>
<td>state space of an $N = (1, 1)$ superconformal extension of $\mathcal{Q}$, see Sect. 7.6 below</td>
</tr>
<tr>
<td>Hilbert space $\mathcal{H}_{e-p}$ of square-integrable differential forms on $G$</td>
<td>Ramond generators $G_0, \overline{G}_0$ of Sect. 7.6 below</td>
</tr>
<tr>
<td>Operators $\mathcal{D}, \overline{\mathcal{D}}$ defined in eq. (4.76)</td>
<td></td>
</tr>
</tbody>
</table>
Since the parameter space-time of a conformal field theory $Q$ is the cylinder $\Sigma$, it is plausible that the spectral data $(F, H, H)$ of $Q$ describe the (non-commutative) geometry of some loop space $M^{S^1}$, where, for a rational conformal field theory, “target space” $M$ can be expected to be some compact (non-commutative) space. In other words, a two-dimensional conformal field theory is always a conformal, non-linear $\sigma$–model of maps $X : \Sigma \to M$, but the target space $M$ might, in general, be a non-commutative space in the sense of Connes [5]. This is the claim advanced and partially substantiated in [78,24].

In fact, the local fields $\varphi_{\chi_j}^{\alpha \beta}(\sigma, \tau = 0)$ — which, unfortunately, are not well-defined — could be interpreted as “functions on loop space $M^{S^1}$” in a natural way. To see this, we consider a compact, smooth classical manifold, $M$. When equipped with Tychonov’s topology, loop space $M^{S^1}$ is a compact Hausdorff space. The Stone-Weierstrass theorem then says that any set of continuous functions on $M^{S^1}$ that separate points in $M^{S^1}$ (i.e., that distinguish two arbitrary loops $X_1$ and $X_2$ in $M$) is total (i.e., spans a dense set in $C(M^{S^1})$). Clearly, $C(M)$ separates points of $M$. Let $\varphi$ be an element of $C(M)$. Then $\varphi$ determines a continuous function $\varphi_{\sigma}$ on $M^{S^1}$ defined by

$$
\varphi_{\sigma}(X) := \varphi(X(\sigma))
$$

\forall X \in M^{S^1} \text{ (i.e., } X : S^1 \to M).$$

The set

$$\{ \varphi_{\sigma} | \varphi \in C(M), \ 0 \leq \sigma < 2\pi \}$$

separates points in $M^{S^1}$ and hence is total in $M^{S^1}$.

It is tempting to identify the local fields $\varphi_{\chi_j}^{\alpha \beta}(\sigma, 0)$ with the functions $\varphi_{\sigma}$, for a suitable choice of the target space $M$. This interpretation would be particularly natural in the examples of the two-dimensional Wess-Zumino-Witten models [41,42]: Let $G$ be a simply-laced compact Lie group. Let $\widehat{g}_k$ be the corresponding Kac-Moody current algebra at positive integer level $k$. The WZW model based on the group $G$ at level $k$ is defined by setting

$$E = \bar{E} = U(\widehat{g}_k).$$

(7.134)

In this model, the primary local fields $\varphi_{\chi_j}^{\alpha \beta}(\sigma, \tau)$ are given by

$$\varphi_{\chi_j}^{\alpha \beta}(\sigma, \tau) = j(g(\sigma, \tau))_{\alpha \beta},$$

(7.135)

where $j$ is an “integrable” representation of $G$ (see e.g. [24]), $g(\cdot, \tau)$ denotes a loop in $G$, i.e., an element of $G^{S^1}$, and $j(g)_{\alpha \beta}$ denotes the $\alpha \beta$ matrix element of $j(g)$.

This example is interesting in two respects:

(1) In general, only the smeared field operators

$$\int_{-\infty}^{\infty} d\tau f(\tau) j(g(\sigma, \tau))$$

are well-defined operators on the Hilbert space $\mathcal{H} = \mathcal{H}_{G,k}$ of the model, where $f$ is an arbitrary Schwartz space test function (see (7.132)). Thus, only the algebra $\mathcal{F}$ of “functions on quantized phase space over $G^{S^1}$ ” makes sense, rather than the analogue of $C(G^{S^1})$. 

116
(2) Given \( G \) and \( k < \infty \), the list \( \mathcal{I} \) of “integrable” representations of \( G \), which correspond to the irreducible unitary positive-energy representations of \( \mathcal{E} \cong U(\mathfrak{g}_k) \), is finite. Thus, 

\[
\left\{ j(g)_{\alpha\bar{\beta}} \mid j \in \mathcal{I} \right\} \cong \bigoplus_{j \in \mathcal{I}} W_j \otimes W_j^\vee
\]

is not nearly dense in \( C(G) \).

We conclude that if the WZW model based on the group \( G \) at level \( k < \infty \) describes the geometry of some loop space \( M^{S^1} \) then \( M \) cannot be the group manifold of \( G \). It turns out (see [24,106,107]) that \( M \) can be interpreted as a non-commutative space corresponding to a quantum deformation of \( G \), described by the property that the “algebra of functions on \( M \)” is the “algebra of functions on a quantum group” corresponding to \( G \), with deformation parameter

\[
q = \exp i\pi / (k + g^\vee)
\]

where \( g^\vee \) is the dual Coxeter number of \( g \).

The target space geometry of the WZW model (with \( G = SU(2) \)) has been studied in some detail in [24,106,107]. We shall outline the results in the next subsection.

### 7.5 Reconstruction of (non-commutative) target spaces from conformal field theory

In the last subsection, we have argued that two-dimensional conformal field theory describes loop space geometry. We have encountered the technical problem that the local fields \( \varphi_{X_j \otimes X_j}^\sigma(\sigma, \tau = 0) \), which, formally, can be identified with functions on loop space \( M^{S^1} \) of some target space \( M \), are not well-defined operators on the Hilbert space \( \mathcal{H} \) of state vectors of the conformal field theory. However, the field operators \( \varphi_{X_j \otimes X_j}^\sigma(\sigma, f) \) smeared in parameter-time \( \tau \), defined in (7.132), are well-defined operators on \( \mathcal{H} \) and generate a non-abelian \( \ast \)-algebra \( \mathcal{F} \) of “functions on quantized phase space” of \( M^{S^1} \).

Unfortunately, there are no local fields \( \varphi_{X_j \otimes X_j}^\sigma(\sigma, \tau) \) that are independent of \( \sigma \); i.e., a priori, there are no candidates for “functions on constant loops” from which one could construct an algebra of “continuous functions” on target space \( M \). The algebra \( \mathcal{F} \) does not contain any \( \ast \)-subalgebra that could be interpreted as the algebra of continuous functions on \( M \); worse: there does not appear to exist any non-trivial \( \ast \)-homomorphism from \( \mathcal{F} \) to some \( C^\ast \)-algebra that could be interpreted as the algebra of continuous functions on \( M \). (Note that e.g. in the \( \lambda \varphi^4 \) theory, where \( \varphi \) is a real-valued scalar field in \( 1 + 1 \) dimensions, one can define the commutative \( \ast \)-algebra generated by all operators \( \varphi(f, \tau = 0), f \) an arbitrary test function, which describes loop space over \( \mathbb{R} \). In a conformal field theory, it is, in general, not possible to multiply fields smeared out with test functions \( f(\tau) \), at fixed \( \tau \), simply because the scaling dimensions of the fields are too large.)

Thus, in order to reconstruct the target space \( M \) from spectral data \( (\mathcal{F}, \mathcal{H}, H) \) of some conformal field theory \( \mathcal{Q} \), one needs some new ideas. In the following, we briefly review such ideas, as proposed in [78,24,106,107].
(1) We start by identifying vector fields on $M$. From eq. (7.115) in subsection 2), it is clear that the (anti-)chiral algebras $E(\hat{E})$ of conformal field theories $Q$ are always $\mathbb{Z}$-graded. The grading operator is the generator $L_0$ of Vir $\subseteq E$. Let $\{\psi_n^{(K)} \mid n \in \mathbb{Z}, \ K \in \mathcal{K}_+\}$ be a system of $W$-algebra generators of $E$ with conformal weights $h_K \geq 0$; i.e., $E$ is spanned linearly by the $\psi^{(K)}$ and their normal ordered products (see e.g. [111]). For example, if $E$ is the universal enveloping algebra of some Kac-Moody current algebra based on a semi-simple Lie algebra $g$ then

$$\psi_n^{(K)} = J^A_n, \ A = 1, \ldots, \dim g,$$

where $J^A_n$ are the modes of chiral currents, $J^A(\sigma + \tau)$, of conformal weight $h_A = 1$.

By eq. (7.115),

$$[L_0, \psi_n^{(K)}] = -n \psi_n^{(K)}, \ n \in \mathbb{Z}.$$ We define $E^{(0)}$ to be the *-subalgebra of $E$ generated by

$$\{\psi_n^{(K)} \mid n \in \mathbb{Z}, \ |n| \leq h_K - 1, \ K \in \mathcal{K}_+\}.$$ Similarly, $\hat{E}^{(0)}$ is the *-algebra generated by

$$\{\chi_n^{(K)} \mid n \in \mathbb{Z}, \ |n| \leq \bar{h}_K - 1, \ K \in \mathcal{K}_-\}.$$ It is tempting to interpret the generators (7.139,140) as vector fields on the target space, $M = M_Q$, of the conformal field theory $Q$. If $Q$ is a WZW model based on the Lie algebra $g$ of a compact semi-simple Lie group $G$ then $h_K = \bar{h}_K = 1$, for all $K \in \mathcal{K}_+$ and $\bar{K} \in \mathcal{K}_-$, and the generators (7.139) and (7.140) can indeed be identified with left- and right-invariant vector fields on $G$, respectively.

An important observation is that the co-products $\Delta_z$ map $E^{(0)}$ to $E^{(0)} \otimes E^{(0)}$, as is implied by eq. (7.111). In particular, if $E = U(\hat{g}_k)$ then $E^{(0)} = U(g)$, and the co-products $\Delta_z |_{E^{(0)}}$ coincide with the usual co-product of $U(g)$.

(2) Given a highest weight representation $j \in I$ of $E$, let $\mathfrak{h}_{j,0}$ be the subspace of the representation space $\mathfrak{h}_j$ spanned by all highest weight vectors in $\mathfrak{h}_j$ (in the sense of eq. (7.117)). We define $\mathfrak{h}_j^{(0)}$ to be the closure of the subspace of $\mathfrak{h}_j$ spanned by

$$\{a\chi \mid \chi \in \mathfrak{h}_{j,0}, \ a \in E^{(0)}\}.$$ In the example of a WZW model based on a compact, semi-simple Lie group $G$ with Lie algebra $g$, $I = \bar{I}$ consists of all integrable representations of $g$, $E^{(0)} \cong U(g)$, and $\mathfrak{h}_j^{(0)}$ is the finite-dimensional representation space for the representation $j \in I$ of $g$.

(3) We are now prepared to define an “algebra of continuous functions” $A_Q$ on the target space $M_Q$ of a rational conformal field theory $Q$. There are (at least) the following two ways of defining $A_Q$, which, in the example of WZW models, are expected to be equivalent (and related to quantum group theory; see [106]).

(3.1) In the first approach, we take $A_Q$ to be “generated” (in a sense made precise below) by primary local fields $\varphi_{\lambda_j}(\sigma, \tau)$ of $Q$ as defined below eq. (7.129), which are labeled by certain pairs of representations $(j, \bar{j})$ of $E \times \hat{E}$ ranging over some subset $\Pi \subseteq I \times \bar{I}$. In
restricting to primary fields, we again draw inspiration from the example of WZW models, or, put differently, from string theory on group manifolds: In this example, the primary fields can be regarded as functions on the (deformed) group manifold or as functions of “centre of mass” coordinates of the moving string, while descendant fields would describe excited string oscillations around the centre of mass.

Moreover, the example of a string moving in a toroidal target tells us that it might be appropriate to restrict the “generators” of \( \mathcal{A}_Q \) further by only using primary fields \( \varphi^\alpha_{\chi_j \otimes \chi_j} \) with \((j, j) \in \Pi(0)\), where \( \Pi(0) \) is a subset of \( \Pi \) containing the label \((j_0, j_0)\) of the field of lowest non-trivial scaling dimension \( d_0 = h_{j_0} + h_{j_0} \) such that \( h_{j_0} = h_{j_0} \), together with those of all primaries that arise from repeated OPE of this field with itself. (More precisely, we require that \( \Pi(0) \) is a sub-ring of the fusion ring.) If, in the case with toroidal targets, we were to include all primary fields as “generators” of \( \mathcal{A}_Q \), we would obtain functions depending not only on the usual Fourier (“momentum”) modes, but also on the winding number. This would, make it impossible to distinguish the torus from its \( T \)-dual.

To give a precise meaning to the algebra \( \mathcal{A}_Q \), we define a subspace \( \hat{\mathcal{H}}_Q^{(0)} \) of the Hilbert space \( \mathcal{H}_Q \) of \( Q \) (defined in eq. (7.129)) by setting

\[
\hat{\mathcal{H}}_Q^{(0)} = \bigoplus_{(k, \bar{k}) \in \Pi(0)} b_k^{(0)} \otimes b_{\bar{k}}^{(0)} \otimes \mathbb{C}^{n(k, \bar{k})},
\]

where \( b_{(k, \bar{k})} \) is as in (7.141). Obviously, \( \hat{\mathcal{H}}_Q^{(0)} \) is invariant under \( \mathcal{E}^{(0)} \otimes \overline{\mathcal{E}}^{(0)} \) and under the Virasoro generators \( H = L_0 + \bar{L}_0 \), \( P = L_0 - \bar{L}_0 \), therefore we can restrict to the zero-momentum subspace

\[
\mathcal{H}_Q^{(0)} := \{ \chi \in \hat{\mathcal{H}}_Q^{(0)} \mid P \chi = 0 \}.
\]

With each pair \((\chi_k, \chi_{\bar{k}})\) of vectors in \( b_k^{(0)} \times b_{\bar{k}}^{(0)} \), for \((k, \bar{k}) \in \Pi(0)\), and each local field \( \varphi^\beta_{\chi_k \otimes \chi_{\bar{k}}} (z, \bar{z}) \) of the conformal field theory \( Q \), we associate an element \( \Phi^\beta_{\chi_k \otimes \chi_{\bar{k}}} \) of the algebra \( \mathcal{A}_Q \), which we define in terms of its matrix elements between a total set of vectors in \( \mathcal{H}_Q^{(0)} \): For \( i = j \) or \( l \), \( \chi_i \otimes \chi_l \otimes u_\alpha \in b_i^{(0)} \otimes b_l^{(0)} \otimes \mathbb{C}^{n(i, l)} \subset \mathcal{H}_Q^{(0)}, (i, \bar{l}) \in \Pi(0) \), we define

\[
\langle \chi_j \otimes \chi_j \otimes u_\alpha \mid \Phi^\beta_{\chi_k \otimes \chi_{\bar{k}}} \mid \chi_l \otimes \chi_{\bar{l}} \otimes u_\gamma \rangle := C \left( \chi_j, \chi_j, \alpha \right)_{\chi_k, \chi_{\bar{k}}, \beta} \left( \chi_l, \chi_{\bar{l}}, \gamma \right)_{\mathcal{H}_Q},
\]

where the \( C \)'s are the coefficients in the operator product expansion (7.131). This is the proposal made in ref. [24]. Because the local fields \( \varphi^\beta_{\chi_k \otimes \chi_{\bar{k}}} (z, \bar{z}) \) form a \( * \)-algebra, the algebra \( \mathcal{A}_Q \) generated by \( \{ \Phi^\beta_{\chi_k \otimes \chi_{\bar{k}}} \mid \chi_k \otimes \chi_{\bar{k}} \otimes u_\beta \in \mathcal{H}_Q^{(0)} \} \) is a \( * \)-algebra of operators represented on \( \mathcal{H}_Q^{(0)} \). If \( \mathcal{H}_Q^{(0)} \) is finite-dimensional, an assumption that holds e.g. for the WZW models (at finite level), then the operators \( \Phi^\beta_{\chi_k \otimes \chi_{\bar{k}}} \) are bounded and \( \mathcal{A}_Q \) is a direct sum of full matrix algebras.

It is crucial to observe that, by definition (7.144),

\[
\left[ \psi_n^{(K)}, \Phi^\beta_{\chi_k \otimes \chi_{\bar{k}}} \right] = \Phi^\beta_{k \delta (\psi^{(K)})_n} \chi_k \otimes \chi_{\bar{k}}
\]

for all generators \( \psi_n^{(K)} \) of \( \mathcal{E}^{(0)} \), with \( \delta_1 (\psi^{(K)})_n \in \mathcal{E}^{(0)} \), and that

\[
\left[ \overline{\psi}_n^{(K)}, \Phi^\beta_{\chi_k \otimes \chi_{\bar{k}}} \right] = \Phi^\beta_{\chi_k \otimes k \delta (\overline{\psi}^{(K)})_n} \chi_k \otimes \chi_{\bar{k}}.
\]

119
for all generators \( \tilde{\psi}^{(K)}_n \) of \( \mathcal{E}^{(0)} \), with \( \delta_1(\tilde{\psi}^{(K)})_n \in \mathcal{E}^{(0)} \). This follows quite easily from (7.139–142,144) and the definition (7.111) of the co-products \( \Delta_z \). In particular, in the example of a WZW model based on a compact, semi-simple Lie group \( G \) with Lie algebra \( \mathfrak{g} \), we have that

\[
\begin{bmatrix}
J^A \\
\Phi_{\chi_k}^\beta \otimes \chi_k
\end{bmatrix} = \Phi_{k(J^A)\chi_k}^\beta,
\]

where \( \{J^A \equiv J^A_{n=0}\} \) is a basis of \( \mathfrak{g} \), and

\[
\begin{bmatrix}
\tilde{J}^A \\
\Phi_{\chi_k}^\beta \otimes \chi_k
\end{bmatrix} = \Phi_{\tilde{k}(J^A)\chi_k}^\beta.
\]

We concentrate on the WZW model with diagonal modular invariant; then \( \Pi^{(0)} \) is itself given by the diagonal in \( I \times \tilde{I} \), i.e., \( k \equiv \tilde{k} \) for all \( (k, \tilde{k}) \in \Pi^{(0)} \), and \( \beta \) has only a single value (and hence can be omitted). Eqs. (7.147) and (7.148) are analogous to the two intertwining relations stated in Sect. 4.2, below eq. (4.82). When the level of the WZW model tends to \(+\infty\) (i.e., in the “classical limit”), eqs. (7.147) and (7.148) become equivalent to those intertwining relations!

In accordance with our discussion in Section 2, below eq. (2.9), an algebra \( \mathcal{F}^{(0)}_Q \) of “functions on quantized phase space” over target space \( M_Q \) can be defined as the \( * \)-algebra of operators on \( \mathcal{H}^{(0)} \) generated by \( \mathcal{A}_Q \), \( \mathcal{E}^{(0)} \) and the Hamiltonian \( H \).

The metric non-commutative geometry of the target space \( M_Q \) is thus encoded in the spectral data

\[
(\mathcal{A}_Q, \mathcal{H}^{(0)}, H) \quad \text{and} \quad (\mathcal{F}^{(0)}_Q, \mathcal{H}^{(0)}, H).
\]

In the example of the SU(2)–WZW model at integer level \( k \), the non-commutative Riemannian geometry of the “fuzzy three-sphere” described by \( (\mathcal{A}_Q^{(0)}, \mathcal{H}^{(0)}, H) \) has been studied quite explicitly in [107].

(3.2) There is an alternative definition of the algebra \( \mathcal{A}_Q \) of “functions on \( M_Q \)” studied in [106]: It is based on the idea that the target space will, to a large extent, be determined by its (quantum) symmetries. One defines \( \mathcal{A}_Q \) to be the (generally non-associative) \( * \)-algebra generated by operators

\[
\left\{ \phi_{\chi_{\bar{k}}}^\beta \otimes \chi_k \mid \chi_k \in \mathfrak{b}_k^{(0)}, \chi_{\bar{k}} \in \mathfrak{b}_{\bar{k}}^{(0)}, (k, \bar{k}) \in \Pi^{(0)}, \alpha = 1, \ldots, n (k, \bar{k}) \right\}
\]

with multiplication table given by

\[
\phi_{\chi_{\bar{j}}}^\alpha \otimes \chi_{\bar{j}} \ast \phi_{\chi_k}^\beta \otimes \chi_k = \sum C(\chi_j, \chi_{\bar{j}}, \alpha \mid \chi_k, \chi_{\bar{k}}, \beta \mid \chi_l, \chi_{\bar{l}}, \gamma) \phi_{\chi_l}^\gamma \otimes \chi_{\bar{l}}.
\]

This approach may help to clarify the connections of non-commutative target space geometry to quantum group theory. But we shall not pursue it here.

In order to study the cohomology of non-commutative target spaces and their Riemannian or complex non-commutative geometries, we should study supersymmetric extensions of conformal field theory. This is the subject of the following last subsection.

In the example of the SU(2)–WZW model at level \( k \), the coefficients \( C(\cdot \mid \cdot \cdot \cdot) \) in eqs. (7.130), (7.144), (7.150) have been calculated explicitly: Let \( \{\chi_i^a\} \) be an orthonormal basis in the representation space \( W_s \) of SU(2) of spin \( s \leq \frac{k}{2} \). Then

\[
C(\chi_{s_1}^i, \chi_{s_1}^{\bar{i}} \mid \chi_{s_2}^j, \chi_{s_2}^{\bar{j}} \mid \chi_{s_3}^l, \chi_{s_3}^{\bar{l}}) = C^k(s_1, s_2, s_3) C^k(\bar{s}_2, s_3, \bar{s}_1)_{s_1}^{j_{\bar{i}}},
\]

120
where the tensors $C_{jli}(s_2,s_3|s_1)$, $\bar{C}_{jli}$ (proportional to squares of Clebsch-Gordan matrices) are defined in (4.82), and the coefficients $C^k(s_1,s_2,s_3)$ enforce the $\hat{su}(2)_k$ fusion rules; explicit expressions for $C^k(s_1,s_2,s_3)$ may be found in [24] and refs. given there. It is shown in [107] that, in this example, the algebra $\mathcal{A}_{\text{SU}(2)-\text{WZW}}$ of “functions” over $M$ is a full matrix algebra; the same is true for the algebra $\mathcal{F}_{\text{SU}(2)-\text{WZW}}$ of “functions on quantized phase space” over $M$. In [106] some steps are undertaken to reconstruct the conformal field theory $\mathcal{Q}$ from the data $(\mathcal{A}_\mathcal{Q}, \mathcal{H}^{(0)}, H)$ together with $\mathcal{E}^{(0)}, \bar{\mathcal{E}}^{(0)}$, in the example of WZW models. This program makes contact with the theory of lattice Kac-Moody algebras.

Clearly, the program sketched in this subsection to reconstruct the (generally non-commutative) target spaces of conformal field theories remains tentative and must be tested in some interesting examples. First steps in this direction have been taken in [24,107,106]. Examples that are reasonably well understood involve chiral algebras obtained from Kac-Moody current algebras or from coset constructions based on Kac-Moody algebras [24]. Whereas WZW models based on compact semi-simple Lie groups describe non-commutative targets, those built on direct products of $\text{U}(1)$–current algebras yield target spaces which are tori. In these examples, dual tori are identified, thanks to our choice of the set $\Pi^{(0)}$ in the definition of $\mathcal{A}_\mathcal{Q}$. They are the simplest examples for $T$-duality. Moreover, the $\text{U}(1)$–models also provide simple examples of mirror symmetry [110]; see [24].

Superconformal field theories of considerable interest in string theory would be the Gepner models, whose target spaces are expected to correspond to (non-commutative deformations of) Calabi-Yau spaces. They remain to be understood more precisely.

In attempting to reconstruct target spaces of conformal field theories one finds that the definition of the algebra $\mathcal{A}_\mathcal{Q}$ of “functions on target space” usually involves considerable arbitrariness. This arbitrariness is at the origin of $T$–duality and of mirror symmetry. The latter is related to an arbitrariness in the definition of the degree of field operators of $N = (2,2)$ superconformal field theories (cf. the next section) and to the fact that, usually, there are several options for choosing $\mathcal{A}_\mathcal{Q}$ as a subalgebra of an algebra $\mathcal{F}_\mathcal{Q}^{(0)}$ of “functions on quantized phase space”, for a given conformal field theory $\mathcal{Q}$. These issues deserve further study.

Let us add an observation on the nature of the “full” target space-time of a string theory reconstructed by the scheme outlined above: After separation of variables, as in (7.29,55), and identifying the algebra of functions on “internal space” with the algebra $\mathcal{A}_\mathcal{Q}$ — which typically is a finite-dimensional matrix algebra — we are led to targets described by algebras of the form $C^\infty(M^4) \otimes \mathcal{A}_\mathcal{Q}$ which resemble the space-times underlying the Connes-Lott construction of the Standard Model!

### 7.6 Superconformal field theories, and the topology of target spaces

In this subsection, we consider conformal field theories whose chiral and anti-chiral algebras $\mathcal{E}$ and $\bar{\mathcal{E}}$ have $\mathbb{Z}_2$–graded extensions $\mathcal{C}_\mathcal{R}$ and $\mathcal{C}^\#_\mathcal{S}$, as discussed in subsection 2) after eq. (7.109), i.e., $\mathcal{C}_\mathcal{R}$ and $\mathcal{C}^\#_\mathcal{S}$ contain fermionic (odd) generators besides the bosonic ones. We demand that $\mathcal{C}^\#_\mathcal{S}$ contain a super-Virasoro algebra. Examples are the supersymmetric WZW models (see [24] and refs. given there). Superconformal field theories realize the
There are three supersymmetric extensions of the Virasoro algebra that are important in the study of superstring vacua: the $N = 1, 2$ and 4 super-Virasoro algebras.

1) The $N = 1$ super-Virasoro algebra

It has generators $\{ L_n \}_{n \in \mathbb{Z}}$ and $\{ G_r \}_{r \in \mathbb{Z} + \frac{1}{2}}$ satisfying the commutation relations

\[
[L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} n (n^2 - 1) \delta_{n+m,0}, \tag{7.152}
\]

\[
[L_n, G_r] = \left( \frac{n}{2} - r \right) G_{n+r},
\]

\[
\{ G_r, G_s \} = 2 L_{r+s} + \frac{c}{3} \left( r^2 - \frac{1}{4} \right) \delta_{r+s,0}.
\]

On the Ramond sector (periodic b.c.), the indices $r$ of the “Ramond generators” $G_r$ range over $\mathbb{Z}$, while, on the Neveu-Schwarz sector (anti-periodic b.c.), the indices $r$ of the “Neveu-Schwarz generators” $G_r$ range over $\mathbb{Z} + \frac{1}{2}$.

In a unitary representation, we have that

\[
L_n^* = L_{-n}, \quad G_r^* = G_{-r}. \tag{7.153}
\]

An $N = (1, 1)$ supersymmetric conformal field theory has the property that both $C_*$ and $\bar{C}_*$ contain an $N = 1$ super-Virasoro algebra (and we shall assume, for simplicity, that $C_* \cong \bar{C}_*$). Then the operators

\[
\mathcal{D} := G_0, \quad \mathcal{D} := \bar{G}_0 \tag{7.154}
\]

play the role of the two Pauli-Dirac operators in $N = (1, 1)$ supersymmetric spectral data, and

\[
d := G_0 - i \bar{G}_0, \quad d^* := G_0 + i \bar{G}_0 \tag{7.155}
\]

can be interpreted as exterior derivative and its adjoint in spectral data with centrally extended $N = (1, 1)$ supersymmetry, as studied in subsection 8) of Sect. 5.2.

We define

\[
\lambda_n := L_n - \bar{L}_{-n}, \tag{7.156}
\]

and

\[
d_n := \frac{2}{n} [\lambda_n, d] = G_n - i \bar{G}_{-n}.
\]

If the central charges $c$ and $\bar{c}$ of the two Virasoro algebras in $C_R$ and $\bar{C}_R$ coincide, as assumed above, then we have that

\[
[\lambda_n, \lambda_m] = (n - m) \lambda_{n+m} \quad \text{(Witt algebra)},
\]

\[
[\lambda_n, d_m] = \left( \frac{n}{2} - m \right) d_{n+m},
\]

and

\[
\{ d_n, d_m \} = 2 \lambda_n.
\]

These commutation relations are the structure relations of the super-Witt algebra, compare to Sect. 5.3, eqs. (7.77, 7.78).
If $\mathcal{F}_Q$ denotes the field algebra (algebra of “functions on quantized phase space”) of an $N = (1, 1)$ superconformal field theory $Q$, as defined below eq. (7.132) of subsection 2), then the spectral data of $Q$ on the Ramond sector are given by

$$(\mathcal{F}_Q, \mathcal{H}_{\text{Ramond}}, \mathcal{D}, \overline{\mathcal{D}}, \gamma, \overline{\gamma}) ,$$

(7.157)

where $\gamma^\# = (-1)^F\#$ and $F\#$ counts the number of left (resp. right) moving fermions. The (anti-)chiral algebras $\mathcal{E}\#$ of $Q$ play the role of reparametrization symmetries. $N = (1, 1)$–superconformal field theories are perfect examples for the mathematical structure described in subsection 8) of Sect. 5.2 (see also Sect. 5.3): There, we have reviewed the topological information encoded in the spectral data (7.157). For example, index theory for a superconformal field theory is concerned with the calculation of the following elliptic genera: the Euler characteristics

$$\chi = \text{tr}_{\mathcal{H}_{\text{Ramond}}} \left( \gamma \otimes \overline{\gamma} e^{i(\tau \mathcal{D}^2 - \overline{\tau} \overline{\mathcal{D}}^2)} \right) ,$$

(which is independent of $\tau, \overline{\tau}$) and the signature genus

$$\Phi(\sigma) = \text{tr}_{\mathcal{H}_{\text{Ramond}}} \left( (1 \otimes \overline{\gamma}) e^{i(\tau \mathcal{D}^2 - \overline{\tau} \overline{\mathcal{D}}^2)} \right) .$$

This genus and the $\hat{A}$ genus are modular forms; for details see [109] and refs. given there. See also eqs. (5.76,77).

The de Rham-Hodge theory has been outlined in subsections 7) and 8) of Sect. 5.2. The Ramond ground states of $Q$, i.e., the highest weight vectors $\psi \in \mathcal{H}_{\text{Ramond}}$ satisfying

$$\mathcal{D} \psi = \overline{\mathcal{D}} \psi = (L_0 - \bar{L}_0) \psi = 0 ,$$

(7.158)

can be interpreted as harmonic forms on the target space $M_Q$ of $Q$; see also [15,16] — unless supersymmetry is spontaneously broken, as described e.g. in Sect. 4.2, eqs. (4.76,77), and in subsection 7) of Sect. 5.2, below eq. (5.60). Spontaneous supersymmetry breaking is encountered e.g. in the study of the $N = (1, 1)$ supersymmetric WZW models; see [24] and refs. given there.

When supersymmetry is spontaneously broken there are no Ramond ground states and the cohomology of the complex of vector forms, see subsection 7) of Sect. 5.2, eqs. (5.48–55), is trivial. But this does not imply that the cohomology of the complexes $\Omega_{\mathcal{D}}^\ast(\mathcal{F}_Q)$ , $\Omega_{\mathcal{D}}^\ast(\mathcal{F}_Q)$, $\Omega_{\mathcal{D}}^\ast(\mathcal{F}_Q^{(0)})$ is trivial, too, where $\mathcal{F}_Q$ is the $\ast$–algebra defined below eq. (7.132), and $\mathcal{F}_Q^{(0)}$ is the $\ast$–algebra of “functions on quantized phase space over target space $M_Q$”, as defined above eq. (7.149). The differential $\ast$–algebras $\Omega_{\mathcal{D}}^\ast(\mathcal{F}_Q)$ and $\Omega_{\mathcal{D}}^\ast(\mathcal{F}_Q^{(0)})$ are defined as in subsection 2) of Sect. 5.1. (We are using here that the “small” Ramond Hilbert space $\mathcal{H}_{\mathcal{D}}^{(0)}$, whose definition can be inferred from eqs. (7.141,142), is invariant under $\mathcal{F}_Q^{(0)}$ and under $\mathcal{D}$ (and $\overline{\mathcal{D}}$); see also [24].) The differential $\overline{\mathcal{I}}$–algebras $\Omega_{\mathcal{I}}^\ast(\mathcal{F}_Q)$ and $\Omega_{\mathcal{I}}^\ast(\mathcal{F}_Q^{(0)})$ are defined as in eqs. (5.34–36) of Sect. 5.2. The cohomology rings $H_{\mathcal{D}}^\ast(\mathcal{F}_Q)$, $H_{\mathcal{D}}^\ast(\mathcal{F}_Q^{(0)})$ are defined as in eqs. (5.58,59) of Sect. 5.2.

Superconformal field theories and their “target space geometry” fit perfectly into the framework developed in Section 5! In particular, the $S^1$–equivariant cohomology of $\Omega_{\mathcal{I}}^\ast(\mathcal{F}_Q)$ is determined according to the theory outlined in subsection 8) of Sect. 5.2.
In order to illustrate the general theory, we summarize results for the example where \( Q \) is the supersymmetric SU(2)--WZW model at level \( k = 1, 2, \ldots \). Proofs for the results stated here can be found in [107].

(i) The “small” Ramond Hilbert space, \( \mathcal{H}^{(0)} = \mathcal{H}^{(0)}_{\text{Ramond}} \), is given by

\[
\mathcal{H}^{(0)} = \mathcal{H}^{(0)}_{\text{bos.}} \otimes F^{(0)},
\]

where

\[
\mathcal{H}^{(0)}_{\text{bos.}} := \bigoplus_{s \leq \frac{k}{2}} W_s \otimes W_s, \tag{7.160}
\]

and \( s \equiv s^\vee = 0, \frac{1}{2}, 1, \ldots, \frac{k}{2} \) is the spin of the representation of SU(2) on \( W_s \); furthermore, \( F^{(0)} \) is the representation space for the unique irreducible representation of the Clifford algebra \( Cl(\mathbb{R}^6) \) with six self-adjoint generators \( \{ \psi^A, \bar{\psi}^A \}_{A=1}^3 \) satisfying

\[
\{ \psi^A, \psi^B \} = \delta^{AB}, \quad \{ \bar{\psi}^A, \bar{\psi}^B \} = \delta^{AB}, \quad \{ \psi^A, \bar{\psi}^B \} = 0.
\]

If the direct sum on the r.s. of (7.160) were unrestricted \( \mathcal{H}^{(0)} \) would be the Hilbert space \( \mathcal{H}^{e-p} \) of square-integrable differential forms on \( SU(2) \cong S^3 \); see Sect. 4.2.

(ii) The algebra \( A^{(0)} = A^{(0)}_{\mathfrak{su}(2)_k} \), which coincides with \( F^{(0)} \) in this example, turns out to be a full matrix algebra,

\[
A^{(0)} = F^{(0)} \cong \text{End} \left( \mathcal{H}^{(0)}_{\text{bos.}} \right).
\]

(iii) The Pauli-Dirac operators \( D := G_0 \big|_{\mathcal{H}^{(0)}} \) and \( \bar{D} := \bar{G}_0 \big|_{\mathcal{H}^{(0)}} \) are given by

\[
D = \psi^A \left( J^A - \frac{i}{12} \varepsilon_{ABC} \psi^B \psi^C \right),
\]

\[
\bar{D} = \bar{\psi}^A \left( \bar{J}^A - \frac{i}{12} \varepsilon_{ABC} \bar{\psi}^B \bar{\psi}^C \right),
\]

where \( J^A \) is the 0–mode of the current \( J^A(z) \) generating the left- (right-) Kac-Moody algebra \( \mathfrak{su}(2)_k \), whose enveloping algebra \( \mathcal{E}(\mathcal{E}) \) is the (anti-)chiral algebra of the theory. Formulas (7.162) are a special case of eqs. (4.76), Sect. 4.2: \( i \Gamma^j \rightarrow \psi^A, \quad i \bar{\Gamma}^j \rightarrow \bar{\psi}^A, \quad i T_j \rightarrow J^A, \quad i \bar{T}_j \rightarrow \bar{J}^A, \quad f_{ijk} \rightarrow \varepsilon_{ABC} \).

Thus, the \( N = (1,1) \) spectral data

\[
(\mathcal{A}^{(0)}, \mathcal{H}^{(0)}, D, \bar{D})
\]

describe the non-commutative geometry of the target space \( M_{\mathfrak{su}(2)_k} \), which is the “fuzzy three-sphere” \( S^3_k \).

(iv) The differential \( \ast \)-algebras \( \Omega^\ast_D(\mathcal{A}^{(0)}) \cong \Omega^\ast_D(\mathcal{A}^{(0)}) \) considered in Sects. 5.1, 5.2 turn out to be given by

\[
\Omega^\ast_D(\mathcal{A}^{(0)}) = \bigoplus_{n=0}^3 \Omega^n_D(\mathcal{A}^{(0)}),
\]

124
where $\Omega^0_D(\mathcal{A}^{(0)})$, $n = 0,\ldots,3$, are free $\mathcal{A}^{(0)}$–modules: $\Omega^0_D(\mathcal{A}^{(0)}) = \mathcal{A}^{(0)}$ and

\[
\begin{align*}
\Omega^1_D(\mathcal{A}^{(0)}) &\text{ has dimension 3, with basis } \{1 \otimes \psi^A\}_A=1, \\
\Omega^2_D(\mathcal{A}^{(0)}) &\text{ has dimension 3, with basis } \{1 \otimes \psi^A \psi^B\}_A<B, \\
\Omega^3_D(\mathcal{A}^{(0)}) &\text{ has dimension 1, with basis } \{1 \otimes \psi^1 \psi^2 \psi^3\}.
\end{align*}
\] (7.165)

Thus, every element $\alpha \in \bigwedge^\bullet D(\mathcal{A}^{(0)})$ can be represented uniquely as

\[
\alpha = \alpha_0 \otimes 1 + \alpha_{1,A} \otimes \psi^A + \alpha_{2,A} \otimes \psi^A \psi^1 + \alpha_3 \otimes \psi^1 \psi^2 \psi^3,
\]

where the coefficients $\alpha_n, \alpha_{n,A}$ are elements of $\mathcal{A}^{(0)}$. Integration of forms is given by

\[
\int \alpha = \text{tr}_{\mathcal{H}^{(0)}}(\alpha_0),
\]

and the metric (Hermitian structure) on $\bigwedge^\bullet D(\mathcal{A}^{(0)})$ by

\[
\langle \alpha, \beta \rangle = \alpha_0 \beta_0^* + \frac{1}{2} \alpha_{1,A} \beta_{1,A}^* + \frac{1}{4} \alpha_{2,A} \beta_{2,A}^* + \frac{1}{8} \alpha_3 \beta_3^*.
\] (7.167)

Following Sect. 5.1, one can equip the “cotangent bundle” $\bigwedge^1_D(\mathcal{A}^{(0)})$ with (left- or right-) connections, $\nabla$, and calculate their Riemann-, Ricci- and scalar curvature; see [107].

Next, we report on the cohomology groups of the fuzzy three-sphere following Connes’ definition of cohomology rings, which is suitable for $N=1$ spectral data as considered in Sect. 5.1. We define

\[
\mathcal{A}^{(0)}_R := \bigoplus_{s \leq \frac{1}{2}} \left(1 \big|_{W_s} \otimes \text{End}(W_s)\right);
\]

(compare with eq. (7.160)). A lengthy calculation (see [107]) shows that

\[
H^0(\mathcal{A}^{(0)}) \cong H^3(\mathcal{A}^{(0)}) \cong \mathcal{A}^{(0)}_R,
\]

and

\[
H^1(\mathcal{A}^{(0)}) = H^2(\mathcal{A}^{(0)}) = \{0\}.
\] (7.168)

These results support the interpretation of target spaces of SU(2)–WZW models as “fuzzy three-spheres”.

One can view the target spaces of WZW models based on a compact, semi-simple group $G$ (at finite level $k = 1,2,3,\ldots$) as examples of non-commutative Riemannian spaces and describe them in terms of $N=(1,1)$ spectral data

\[
(\mathcal{A}^{(0)}, \mathcal{H}^{(0)}, d, d^*, \tilde{\gamma}, \ast).
\] (7.169)

One chooses the differential $d$ to be given by the BRST operator corresponding to the representation of $G$ on $\mathcal{H}^{(0)}$, as in eq. (4.86) of Sect. 4.2; the generators $T_j$ in eq. (4.86) are defined in terms of the zero-modes of the left- and/or right-moving Kac-Moody currents of the model. In the example of the SU(2)–WZW-models at level $k$, the “de Rham cohomology groups” (see Sect. 5.2) determined by the BRST operators have the form

\[
H^0(\mathcal{A}^{(0)}) \cong H^3(\mathcal{A}^{(0)}) \neq \{0\}, \quad H^1(\mathcal{A}^{(0)}) = H^2(\mathcal{A}^{(0)}) = \{0\}.
\]
For further details see [107,24,106].

2) $N = 2$ and $N = 4$ supersymmetry; mirror symmetry

In the study of superstring vacua exhibiting space-time supersymmetry one is led to consider superconformal field theories with higher (world sheet) supersymmetries, in particular, with 2 or 4 supersymmetries in each chiral sector; see [29]. Properties of such conformal field theories can be derived from the representation theory of $N = 2$ or $N = 4$ super-Virasoro algebras.

The $N = 2$ super-Virasoro algebra has generators $\{L_n, G_{n+a}^+, J_n\}_{n \in \mathbb{Z}}$, with $a = 0$ on the Ramond sector (periodic boundary conditions), and $a = \frac{1}{2}$ on the Neveu-Schwarz sector (anti-periodic boundary conditions). They satisfy the commutation relations

\[
(i) \quad [L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} n(n^2 - 1) \delta_{n+m,0},
\]

\[
(ii) \quad [L_n, J_m] = -m J_{n+m},
\]

\[
(iii) \quad [J_n, J_m] = \frac{c}{3} n \delta_{n+m,0},
\]

\[
(iv) \quad [L_n, G_{m+a}^+] = \left(\frac{n}{2} - (m + a)\right) G_{n+m+a}^+
\]

\[
(v) \quad [J_n, G_{m+a}^+] = \pm G_{n+m+a}^+
\]

\[
(vi) \quad \{G_{n+a}^+, G_{m+a}^-\} = 2 L_{n+m} + (n - m + 2a) J_{n+m} + \frac{c}{3} \left(n^2 - \frac{1}{4}\right) \delta_{n+m,0}
\]

\[
(vii) \quad \{G_{n+a}^+, G_{m+a}^+\} = \{G_{n+a}^-, G_{m+a}^-\} = 0.
\]

In a unitary representation of the $N = 2$ super-Virasoro algebra,

\[
L_n^* = L_{-n}, \quad J_n^* = J_{-n}, \quad (G_{n+a}^+)^* = G_{-n-a}^-.
\]

Relations (7.170) and (7.171) show that the generators $G_0^+$ and $G_0^-$ of an $N = 2$ super-Virasoro algebra correspond to the operators $\partial = D_1 - i D_2$ and $\partial^* = D_1 + i D_2$ of $N = 2$ spectral data, as discussed in subsection 8) of Sect. 5.1, eqs. (5.24, 5.25). The zero mode $J_0$ of the current $J$ corresponds to the $\mathbb{Z}$-grading operator $T$ of eq. (5.26). Thus, the $N = 2$ super-Virasoro algebra is related to spectral data describing Kähler geometry. The Virasoro subalgebra with generators $\{L_n\}_{n \in \mathbb{Z}}$ plays the rôle of a Lie algebra $\mathcal{G}$ of infinitesimal reparametrizations, as discussed at the beginning of Sect. 5.3; see Definition A of Sect. 5.3.

A local, unitary, “left-right symmetric” superconformal field theory with $N = (2,2)$ supersymmetry on the Ramond sector $\mathcal{H}_R$ is given in terms of spectral data

\[
\left(\mathcal{F}, \mathcal{H}_R, \partial, \partial^*, \overline{\partial}, \overline{\partial}^*, T, \overline{T}, \mathcal{G}, \overline{\mathcal{G}}\right)
\]

(7.172)

where $\mathcal{F}$ is a $^*$-algebra of operators on $\mathcal{H}_R$ constructed from local bosonic fields of the theory, $\partial = G_0^+$, $\partial^* = G_0^-$ are the zero-modes of the left-moving Ramond generators, $\overline{\partial} = \overline{G}_0^\pm$, $\overline{\partial}^* = \overline{G}_0^\mp$ — see below for the choice of sign — are the zero modes of the right-moving Ramond generators, $T = J_0$ is the grading operator of the left-movers, while $\overline{T} = \pm J_0$ is the grading operator of the right-movers (but the spectra of $J_0$ and $\overline{J}_0$ are not, in general, contained in the integers), $\mathcal{G}$ and $\overline{\mathcal{G}}$ are Virasoro algebras associated with left and right movers, respectively. The graded Lie algebra $\mathcal{G}_{\partial, \overline{\partial}, T}$ generated by $\partial, \partial^*, T$
and \( \mathcal{G} \) is the \( N = 2 \) super-Virasoro algebra described in (7.170) and (7.171); likewise, the graded Lie algebra \( \overline{\mathcal{G}}_{\mathbb{Z}; \mathbb{R}; \mathcal{T}} \) generated by \( \overline{\partial}, \overline{\partial}^*, \mathcal{T} \) and \( \overline{\mathcal{G}} \) is another copy of the \( N = 2 \) super-Virasoro algebra, and usually the central charges \( c \) and \( \overline{c} \) coincide. Elements of \( \mathcal{G}_{0,0; \mathcal{T}} \) graded-commute with elements of \( \overline{\mathcal{G}}_{\mathbb{Z}; \mathbb{R}; \mathcal{T}} \). This implies, in particular, that \( \{ \partial, \overline{\partial}^* \} = 0 \), i.e., part of the Kähler conditions are satisfied automatically.

By eq. (7.170), (iii) the generators \( \{ J^a_n \}_{n \in \mathbb{Z}} \) form a U(1) current algebra. Setting

\[
G^a_{n+a} := \frac{1}{\sqrt{2}} \left( G^a_{n+a} + G^a_{n+a} \right),
\]

one finds that \( \{ L^a_n, G^a_{n+a} \}_{n \in \mathbb{Z}} \) generate an \( N = 1 \) super-Virasoro algebra, as described in eq. (7.152) above.

In the identifications following eq. (7.172), we have indicated the possibility of two choices of sign in the right moving sector. Indeed, the automorphism

\[
G^\pm_{n+a} \mapsto G^\mp_{n+a} := \overline{G}^\pm_{n+a}, \quad J_n \mapsto J'_n := J_n, \quad \overline{J}_n \mapsto \overline{J}'_n := -\overline{J}_n \tag{7.173}
\]

describes the mirror map, which is a symmetry of the conformal field theory; see [110,112].

From the spectral data (7.172) of an \( N = (2,2) \) superconformal field theory \( \mathcal{Q} \) one can attempt to reconstruct target spaces \( M_\mathcal{Q} \) and \( M'_\mathcal{Q} \) by passing from (7.172) to spectral data

\[
\left( \mathcal{F}_{\mathcal{Q}(0)}, \mathcal{H}_{\mathcal{Q}(0)}, \partial_0, \partial_0^*, \overline{\partial}_0, \overline{\partial}_0^*, T_0, \overline{T}_0 \right),
\]

or

\[
\left( \mathcal{F}^{(0)}_{\mathcal{Q}}, \mathcal{H}^{(0)}_{\mathcal{Q}}, \partial_0, \partial_0^*, \overline{\partial}_0, \overline{\partial}_0^*, T_0, \overline{T}_0 \right), \tag{7.174}
\]

following the constructions in Sect. 7.5 and subsection 1), above: The construction of the algebras \( \mathcal{F}^{(0)}_{\mathcal{Q}} \) and \( \mathcal{F}^{(0)}_{\mathcal{Q}} \), represented on “small Ramond spaces” \( \mathcal{H}^{(0)}_{\mathcal{Q}}, \mathcal{H}^{(0)}_{\mathcal{Q}} \), respectively, (see [24]) involves selecting suitable subrings of local bosonic fields of grade (charge) = (0, 0) indexed by sets \( \Pi^{(0)} \) and \( \Pi^{(0)} \), respectively, of pairs of representations of the chiral algebras, as described in Sects. 7.4 and 7.5. The Dolbeault operators \( \partial_0, \partial_0^*, \overline{\partial}_0, \overline{\partial}_0^* \) are obtained by restricting \( G^+_0, G^-_0, \overline{G}^+_0, \overline{G}^-_0 \) to \( \mathcal{H}^{(0)}_{\mathcal{Q}} \), and \( T_0, \overline{T}_0 \) by restricting \( J_0 \) and \( \overline{J}_0 \) to \( \mathcal{H}^{(0)}_{\mathcal{Q}} \); analogously, the operators \( \partial_0', \partial_0'^*, \overline{\partial}_0', \overline{\partial}_0'^* \) are obtained by restricting \( G^+_0, G^-_0, \overline{G}^+_0, \overline{G}^-_0 \) to \( \mathcal{H}^{(0)}_{\mathcal{Q}} \), respectively, and \( T_0, \overline{T}_0 \) by restricting \( J_0 \) and \( \overline{J}_0 \) to \( \mathcal{H}^{(0)}_{\mathcal{Q}} \), respectively. Thus, the two sets of data in (7.174) are interchanged by the mirror map (7.173), and \( M_\mathcal{Q} \) and \( M'_\mathcal{Q} \) form a mirror pair of (non-commutative) Kähler spaces. Of course, the details of the construction of \( \mathcal{F}^{(0)}_{\mathcal{Q}} \) and \( \mathcal{H}^{(0)}_{\mathcal{Q}} \) or \( \mathcal{F}^{(0)}_{\mathcal{Q}} \) and \( \mathcal{H}^{(0)}_{\mathcal{Q}} \) depend on the superconformal field theory under consideration; a satisfactory, general (model-independent) construction remains to be found. Some simple examples are described in [24].

In spite of the fact that the detailed procedure to reconstruct the (generally non-commutative) target spaces \( M_\mathcal{Q} \) and \( M'_\mathcal{Q} \) is not in general known, at present, a remarkable piece of general theory about \( M_\mathcal{Q} \) and \( M'_\mathcal{Q} \), known: The theory of chiral-chiral and chiral-antichiral rings [112]. In subsection 9) of Sect. 5.2, eq. (5.93) and below, we have defined an algebra

\[
\Omega^{\mathcal{Q}'}_{\partial^*} (\mathcal{A}) = \bigoplus_{p,q} \Omega^{p,q}_{\partial^*} (\mathcal{A}) \tag{7.175}
\]
of Dolbeault forms which is a bi-graded bi-differential algebra. Because $\partial^2 = \overline{\partial}^2 = 0$, $\Omega^\bullet_0$ is a bi-graded complex with respect to graded commutation by $\partial$ and by $\overline{\partial}$.

Setting $A := \mathcal{F}^{(0)}_\Omega$, $\partial := \partial_0$, $\overline{\partial} := \overline{\partial}_0$, as in (7.174), we obtain the differential algebra $\Omega^\bullet_0 \left( \mathcal{F}^{(0)}_\Omega \right)$ of Dolbeault forms on $M_\Omega$. The choice $A := \mathcal{F}^{(0)}_\Omega$, $\partial := \partial_0$, $\overline{\partial} := \overline{\partial}_0$ yields the differential algebra $\Omega^\bullet_0 \left( \mathcal{F}^{(0)}_\Omega \right)$ of Dolbeault forms on the mirror target $M_\Omega$.

Actually, the correct general definition of an algebra $\Omega^\bullet_0 \left( Q \right) = \oplus \Omega^{p,q}_{\partial_0,\overline{\partial}_0} \left( Q \right)$ of Dolbeault forms of an $N = (2,2)$ superconformal field theory $Q$ is to demand that $\Omega^{p,q}_{\partial_0,\overline{\partial}_0} \left( Q \right)$ contain all functionals $\varphi^{p,q}$ of fields of $Q$ with the properties that $\varphi^{p,q}$ leaves $\mathcal{H}^{(0)}_Q$ invariant and

$$[ J_0, \varphi^{p,q} ] = p \varphi^{p,q}, \quad [ \overline{J}_0, \varphi^{p,q} ] = q \varphi^{p,q}. \quad (7.176)$$

In general, the charges (grades) $p$ and $q$ are not integers. However, if $c = \mathfrak{c} = 3n$, $n = 1, 2, 3, \ldots$, and some additional properties are satisfied, $p$ and $q$ turn out to be integers. In this case, a correct choice of the algebra $\mathcal{F}^{(0)}_\Omega$ is one for which

$$\Omega^\bullet_0 \left( Q \right) = \Omega^\bullet_0 \left( \mathcal{F}^{(0)}_\Omega \right). \quad (7.177)$$

An algebra $\Omega^\bullet_0 \left( Q \right)$ is defined similarly, and if $c = \mathfrak{c} = 3n$, one must attempt to choose $\mathcal{F}^{(0)}_\Omega$ such that

$$\Omega^\bullet_0 \left( Q \right) = \Omega^\bullet_0 \left( \mathcal{F}^{(0)}_\Omega \right). \quad (7.178)$$

Eqs. (7.177) and (7.178) are crucial consistency conditions.

The operators $\partial_0$ and $\overline{\partial}_0$ act on $\Omega^\bullet_0 \left( Q \right)$ by graded commutation. One can then attempt to determine the cohomology groups $H^{p,q}_{\partial_0,\overline{\partial}_0} \left( Q \right)$. It turns out that $H^{p,q}_{\partial_0,\overline{\partial}_0} \left( Q \right)$ contains “harmonic forms” $\varphi^{p,q}, \alpha = 1, 2, 3, \ldots$, for all $p, q$, which are in a one-to-one correspondence to chiral-chiral, primary states in the Neveu-Schwarz sector of $Q$. A state $| \varphi^{p,q}_\alpha \rangle$ in the Neveu-Schwarz sector is chiral-chiral iff

$$G_{+1/2}^+ \left| \varphi^{p,q}_\alpha \right\rangle = \overline{G}_{-1/2}^+ \left| \varphi^{p,q}_\alpha \right\rangle = 0 \quad (7.179)$$

and primary iff it is a highest weight vector for the $N = 2$ super-Virasoro algebras. It then follows that $h = \frac{8}{3}, \overline{h} = \frac{8}{3}$, where $h$ and $\overline{h}$ are the conformal weights of $| \varphi^{p,q}_\alpha \rangle$. One can show that $h \leq \frac{3}{2}, \overline{h} \leq \frac{3}{2}$.

It turns out (see [113]) that chiral-chiral primary operators $\{ \varphi^{p,q}_\alpha \}$ form a ring, the chiral-chiral ring $H^\bullet \left( Q \right)$, which, in examples, can often be determined explicitly. If $c = \mathfrak{c} = 3n$ for some positive integer $n$, and assuming that (7.177) holds, then $H^\bullet \left( Q \right)$ is what one might interpret as the Dolbeault cohomology ring of $M_\Omega$.

Analogous results hold when $\partial_0, \overline{\partial}_0$ and $\Omega^\bullet_0 \left( Q \right)$ are replaced by $\partial_0', \overline{\partial}_0'$ and $\Omega^\bullet_0 \left( Q \right)$, respectively. One then arrives at the chiral-antichiral ring $H^\bullet \left( Q \right)$ describing the Dolbeault cohomology ring of $M_\Omega$.

The ring structure of $H^\bullet \left( Q \right)$ and $H^\bullet \left( Q \right)$ is, in general, not that of cohomology rings of classical manifolds, but of certain deformations of such rings, although the dimensions of the spaces of harmonic forms of definite U(1) charge may coincide with the Hodge numbers of a classical Calabi-Yau space.
One can verify that the theory of chiral-chiral and chiral-antichiral rings [112] fits into the general cohomology theory of complex non-commutative geometry, as outlined in Sect. 5.2 and in [18].

For further material on $N = (2, 2)$ superconformal field theories see the lectures by B. Greene.

In the study of string vacua with internal target spaces described by hyper-Kähler manifolds one also encounters $N = (4, 4)$ superconformal field theories. They are based on the representation theory of two copies of the $N = 4$ super-Virasoro algebra with generators $\{L_n, G^A_{r \pm}, T^{I}_m\}$ satisfying the commutation relations

\[
\begin{align*}
(i) & \quad [L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} n(n^2 - 1) \delta_{n+m,0} \\
(ii) & \quad [L_n, T^I_m] = -m T^I_{n+m} \\
(iii) & \quad [T^I_n, T^J_m] = i \varepsilon^{IJK} T^K_{n+m} + \frac{\delta^{IJ}}{12} n \delta_{n+m,0} \\
(iv) & \quad [L_n, G^A_{r \pm}] = \left( \frac{n}{2} - r \right) G^A_{n+r} \\
(v) & \quad [T^I_n, G^A_{r \pm}] = \frac{1}{2} (\sigma^I)^{AB} G^{B+}_{n+r}, \\
& \quad [T^I_n, G^A_{r -}] = -\frac{1}{2} (\sigma^I)^{AB} G^{B-}_{n+r} \\
(vi) & \quad \{G^A_{r +}, G^B_{s -}\} = 2 \delta^{AB} L_{r+s} + 2 (r - s) (\sigma^I)^{AB} T^{I}_{r+s} + \frac{c}{3} \left( r^2 - \frac{1}{4} \right) \delta^{AB} \delta_{r+s,0} \\
(vii) & \quad \{G^A_{r +}, G^B_{s +}\} = \{G^A_{r -}, G^B_{s -}\} = 0.
\end{align*}
\]

Here $A,B \in \{1,2\}$, $I,J,K \in \{1,2,3\}$, $r,s \in \mathbb{Z}$ (Ramond) or $r,s \in \mathbb{Z} + \frac{1}{2}$ (Neveu-Schwarz), and $\sigma^I$, $I = 1,2,3$, are the $2 \times 2$ Pauli matrices. In a unitary representation, one has that

$\begin{align*}
L^*_n = L_{-n}, \quad (T^I_m)^* = T^I_{-m}, \quad (G^A_{r \pm})^* = G^A_{-r \mp}.
\end{align*}$

The operators $\{T^I_n\}_{n \in \mathbb{Z}}$ are the Fourier modes of an $SU(2)$–current $T$ generating an $\tilde{su}(2)$–Kac-Moody algebra at level $k = \frac{c}{8}$. The operators $(G^A_{r \pm}, G^B_{s \mp})$ form $SU(2)$–doublets; thus $SU(2)$ is a “vertical symmetry” of the $N = 4$ super-Virasoro algebra in the sense explained in subsection 9) of Sect. 5.2. It corresponds to the vertical $SU(2)$ symmetry generated by the holomorphic symplectic form and the holomorphic $Z$–grading on the space of holomorphic differential forms on a hyper-Kähler manifold.

Unfortunately, we cannot enter into a more detailed discussion of the mathematically fascinating world of the (non-commutative) target- and loop space geometry of superconformal field theories. We refer the reader to [110,112,24] and the references given there for examples. But we hope that we have made the point that a combination of $N = 2$ (and $N = 4$) superconformal field theory with the methods of non-commutative Kähler and hyper-Kähler geometry, as described in Section 5 and in [18], provides a natural conceptual framework for the study of topics such as mirror symmetry, topology changes, supersymmetric cycles, etc.
8 Conclusions

In these notes we have attempted to review some physical foundations and some conceivably useful mathematical methods that may guide a way towards a “quantum theory of space-time-matter”, yet to be discovered. We have argued (Section 3) that in such a theory, space, time and matter lose their individuality and that classical space-time is an approximate notion that is only appropriate for the description of some asymptotic regimes of a fundamental quantum theory of space-time-matter. The intrinsic geometry of space-time-matter is expected to be non-commutative. This feature can best be taken into account by trying to conceive the fundamental theory as a theory of extended objects. For such a theory to have geometrical content, it is natural to require that its solutions exhibit supersymmetry, i.e., take the form of supersymmetric quantum theories. Key examples of supersymmetric quantum theories are Pauli’s quantum theories of a non-relativistic electron with spin, of its twin, the non-relativistic positron, and of positronium (i.e., of a bound state of an electron and a positron), as described in Section 4. Pauli’s quantum theory of non-relativistic particles with spin neatly encodes the classical differential topology and geometry of Riemannian manifolds and suggests natural generalizations of classical differential topology and geometry, called non-commutative geometry, as described in Connes’ book [5] and in Section 5 and [18]. We have discussed some examples of non-commutative geometrical spaces in Section 6 (non-commutative torus) and Sects. 7.5 and 7.6 (e.g. the “fuzzy 3-sphere”).

First quantized, tree-level superstring theory, as briefly described in Sects. 7.1 and 7.2, is a very sophisticated analogue of Pauli’s quantum theory of non-relativistic, spinning particles. It encodes the topology and geometry of a certain class of loop spaces over generally non-commutative geometrical spaces describing physical space-times. The supersymmetry algebras, more precisely the superconformal field theories, describing superstring vacua provide key tools to explore the geometry of those loop spaces. Unfortunately, space-times described by the vacua of first quantized, tree-level superstring theory are static.

For purposes of physics, the present formulation of first quantized superstring theory is ultimately inadequate in that it is an intrinsically perturbative approach towards understanding the presumably intrinsically non-perturbative quantum dynamics of space-time-matter. It does not appear to enable one to properly describe the dynamical degrees of freedom of non-static, non-commutative quantum space-times.

In order to overcome the shortcomings of first quantized superstring theory, one is tempted to search for “second quantized” theories. In passing from first quantized to second quantized theories, one appears to trade parameter space supersymmetry for target space supersymmetry, and one should worry that one may lose “background independence”. Some preliminary ideas about second quantized, non-perturbative formulations of a quantum theory of space-time-matter have been reviewed in Sect. 7.3 (“matrix models”). They have the positive features that parameter- and target space are treated as non-commutative spaces and that they appear to incorporate some of the general principles reviewed in Section 3. But they have the negative feature that they are based on too rigid a notion of target space (involving global symmetries) and that their very formulation requires choosing a light-cone gauge, so far. In how far superstring theory emerges from matrix models in a limiting regime is only partially understood.

All theories alluded to in Section 7 have the common feature that they yield super-
symmetric spectral data (of the kind studied in Section 5) which enable one to construct non-commutative geometric spaces. While the geometric spaces constructed from the spectral data of vacua of first quantized superstring theory have a more or less direct relationship with space-time, the geometric spaces constructed from spectral data provided by second quantized theories are spaces describing, in principle, all dynamical degrees of freedom (of “space-time-matter”; in a sense they are the configuration- or quantized phase spaces of “space-time-matter”), and it is not clear, yet, how one may extract from them geometrical features of physical space-time.

Yet, the common features of the theories described in Section 7 may encourage us to propose the following

“Geometrization Principle”. A fundamental theory of space-time-matter has solutions yielding supersymmetric spectral data analogous to those described in Section 5 from which models of non-commutative space-time can be reconstructed.

It is likely that the way to finding a satisfactory non-perturbative formulation of a fundamental quantum theory of space-time-matter remains long and steep, resembling an ascent to Mount Everest rather than to Mont Blanc.
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