Paul Dirac and the pervasiveness of his thinking

Invited Lecture presented at the Dirac Centenary Conference, hosted by Baylor University, Texas; 30th September-2nd October 2002

David I Olive

Physics Department, University of Wales Swansea, Swansea SA2 8PP, UK

I shall use a few personal reminiscences of my time as a student and colleague of Dirac in Cambridge to introduce some reflections on the nature of research in theoretical physics. I shall discuss and illustrate the approach of Dirac to his own research and the pervasiveness of the influence his example has provided.

I shall discuss how ideas produced at all stages of his career have proved to be extraordinarily visionary, still motivating and exerting an influence on research many years after his death. Examples include his celebrated equation and prediction of the existence of antiparticles, the magnetic monopole and its quantisation condition, the quantum theory of dynamical systems with constraints and the membrane theory of the electron.
I would like to start by expressing my gratitude to the organisers for inviting me to this fine conference commemorating one of the greatest figures in twentieth century physics. I have to admit that the first time I came across the name of Dirac was when, as a boy, I read a science fiction story about a spaceship powered by a “Dirac drive”. I was given the impression that the “inventor” of this was someone extraordinary but I did not then realise in what way. I have never been able to identify the author or title of this story subsequently.

Many years later, when I came to the University of Cambridge as an affiliated student to study for the mathematics tripos, chance would enroll me at St John’s College, where Dirac was a professorial fellow. By design, I attended the two lecture courses he gave in the academic year 1959-1960, quantum mechanics and advanced quantum mechanics. The next year I became a research student and yet later a research fellow and faculty member in DAMTP. So latterly, and for the four years up until his retirement in 1969, I became a colleague of Dirac, albeit a very junior one, and with little personal contact but enough to realise that the commonly accepted picture of him as pathologically shy and monosyllabic was untrue and unfair.

Like many others, I found his lectures to be a revelation, at last laying out the previously mysterious subject of quantum theory in a comprehensible way, with honesty, beauty and clarity. That year I started attending the weekly seminars in theoretical physics and one of the privileges was to see Dirac take his seat in the front row of the audience. Later, as the seminar progressed, he could often be seen gently lowering his forehead onto the desk in front of him. It was even whispered that he was out of touch with modern developments. To some extent this was true and I think that he was not interested in the analytic properties of the scattering matrix nor the group theoretic classification of particle states, topics which were then so much the centre of attention. Of course what he was doing was simply to follow his own ideas and his own sense of priorities which were illustrated by the subject matter of his advanced lecture course and by seminars he himself gave from time to time. Examples of the latter concerned “The bubble theory of the electron”, “A remarkable representation of the 3+2 de Sitter group” and “Quantum electrodynamics without dead wood”. The first two of these were not mainstream topics at the time but now we are able to see how prescient these interests were and how much he anticipated lines of research which were taken up thirty-five years later in the theory of superstrings and branes, a subject now very fashionable and acknowledged to be the most promising candidate for a truly unified theory. The third topic seemed reactionary but again has, to a large extent, been vindicated by later developments.

The research groups that made up DAMTP were housed together for the first time in 1967 or 1968 when premises between Silver Street and Mill Lane were vacated by the University Press, but Dirac was not assigned an office there as he apparently preferred to work at home. Nevertheless he made a point of attending the morning coffee break once a week in order to distribute preprints he had received and to update us on any relevant gossip. For example, I remember him breaking the tragic news of the death of the Swedish theoretical physicist Gunnar Källén in the crash of his own self-piloted plane.

The only time that I shared a meal with Dirac was at the Director’s table in the restaurant in ICTP in Trieste during a conference on “Renormalisation Theory” in August
1969. On that occasion he was rather communicative and keen to describe an interesting book that he was reading concerning the life and death of Napoleon (It may be relevant that Dirac’s family originated in the Charentes region of eastern France with only a few generations living in Switzerland before settling in England). He was particularly intrigued by the arsenic poisoning that led to Napoleon’s death in exile on the island of St Helena and the theory propounded by the book that this was part of a systematic plot on the part of someone. I have since heard of an alternative to this conspiracy theory whereby the arsenic can be traced to the Paris green dye in the fashionable wallpaper that still hangs in his bedroom there. This dye could not cope with the tropical climate and arsenic leached into the atmosphere.

These minor personal contacts that I had with Dirac indicate to me that he was willing to reveal a very human side to his character. In my experience as a young man at that time this was not so common amongst senior British academics.

I saw Dirac only once after the lunch in Trieste and I cannot resist mentioning this as the occasion was so strange. About ten years after the Trieste encounter I was eating in the restaurant of CERN, which, as you can imagine, was large and crowded, when suddenly at a distance Dirac appeared, unheralded and unannounced, (escorted by J Mehra), and just as suddenly disappeared.

Each year that he was in Cambridge, Dirac gave his lecture course on quantum mechanics based on topics selected from his book, occasionally supplemented by a more advanced course including the remaining topics and also his theory of constrained dynamical systems and their quantisation.

Since Dirac seemed to play little role in departmental affairs he could only exert an influence by example, the example of his exposition of quantum mechanics and that of his many research papers. In all this he demonstrated how it is possible to make every detail of an argument visible and clear in a well-balanced way. Pursuit of logic was a key factor. Followed ruthlessly it led Dirac to the most amazing conclusions and insights.

We all know that he put the finishing touches on the theory and formulation of quantum mechanics adding to its coherence. But more importantly this work provided the foundations for his pioneering work in taking the next steps beyond quantum mechanics to take into account the structure of space-time. He considered the effects of special relativity, what is vaguely called locality, general covariance and even some global topology. I shall not be able to describe his most celebrated achievement, the Dirac equation, what he liked to call “the relativistic wave equation for the electron”, [Dirac 1928] nor the subsequent prediction of antiparticles [Dirac 1931] (I have reviewed this aspect of Dirac’s work elsewhere [Olive 1997]). Nor shall I be able to say much about his procedure of “second quantisation” which relates the principles of quantum mechanics to the structure of space-time and establishes a correspondence between fields and particles [Dirac 1927]. But I do want to to say more about another way in which the notion of locality was developed and I shall do this later on.

There are many ideas about scientific method and the philosophy behind different approaches. The truth is that the success of a method will depend both on the subject matter and the particular scientific worker. It is also true that different methods work better at different stages in the development of the same subject.
Dirac pursued what might be called the aesthetic approach and in his later writing he emphasised the courage required for this sort of work, the courage to pursue an argument to its remorseless conclusion despite the fear that a sudden quirk could well spell disaster. This sort of work requires other qualities that he did not emphasise, for example, a searing honesty, that he so transparently did possess. Besides this, success also requires hard work and dedication, not to mention the sheer genius that no one can explain.

It has to be admitted that the idea of beauty as a criterion for scientific truth is fraught with danger as it can be too subjective and can easily fail in the wrong hands. But it is important to realise that the beauty concerned is not just a snapshot, it also has a quality of overall coherence which I think is important. In the hands of Einstein and Dirac and others the aesthetic approach led to the founding of the cornerstones of physics as known in the first half of the last century.

The initiation of a new era in physics in the middle of the last century was one of the side effects of World War II. Perhaps the main effect was in applied physics but fundamental physics was affected too. Previously, Einstein and Dirac had concentrated their thoughts on the electromagnetic and gravitational fields so important on the macroscopic level. But the frightful effectiveness of the atomic bomb made it clear, even to the general public, that there were other important fields in nature, namely those associated with the structure of the nuclei of atoms. Postwar, new particle accelerators were built to investigate this, and they provided a cornucopia of many, unexpected new elementary particles together with information concerning their properties. Now the aesthetic approach waned in importance as a more pragmatic approach was required to succeed in unravelling the patterns hidden in this data.

This process took about thirty-five years and in the early 1980’s culminated in what is called the “standard model”. This mundane title disguises the fact that this model synthesises a number of extraordinary features that I shall now briefly describe. The model is formulated in terms of a choice of a number of particles (or equivalently their corresponding fields) together with the specification of a system of the non-linear coupled partial differential equations of motion that these fields satisfy (or, more succinctly, by a Lagrangian).

The first surprise is the indirect nature of the good choice of ingredient particles and it is this that partly explains the length of the period of gestation. Neither the proton nor neutron nor any of the newly discovered post-war particles participate. Instead there are two classes in the choice, one augmenting the electron and the other the photon (the quantum of the electromagnetic field). These are the two particles most familiar to prewar physics. The electron class contains the neutrino and strongly interacting counterparts of the electrons called “quarks” out of which the proton and neutron can, in a sense, be constructed. The photon is augmented by other similar particles of unit spin, namely the three weak gauge particles $W^\pm$ and $Z^0$ and the eight colour gluons which couple to themselves and the quarks forming a subtheory called QCD (responsible for strong forces). Out of these extras only the electroweak gauge particles are observed directly (originally at CERN). The quarks are observed only indirectly (originally at SLAC) and the colour gluons even less directly. Actually there are two extra copies of the electron class (called generations) involving the muon and the tau-lepton respectively, together with
extra neutrinos and quarks that occur as constituents of the particles that were discovered postwar. Another ingredient is a set of scalar particles, still being sought, whose fields (the Higgs'), unlike all the others, fail to vanish in the vacuum. The effect of this is what is called a “spontaneous breaking” of the gauge symmetry and a mechanism for providing mass in a natural way.

The second surprise was that, given the good choice of fundamental fields just described, the basic equations that they satisfy are found simply to be elaborations of the familiar equations of Maxwell and Dirac that were already known to describe electrodynamics. Indeed a treatment of this system of equations appears in later editions of Dirac’s textbook on quantum mechanics and was included in the subject matter of his advanced lecture course.

Maybe, after all, it is possible to understand why Dirac used to lower his forehead ever so gently onto his desk. Unfortunately the death of Dirac roughly coincided with the acceptance of the standard model within the particle physics community. Judging by his later writings he remained unhappy with postwar developments in quantum electrodynamics and apparently unaware of the standard model and the extent to which it embodied ideas more familiar to him. But we do know that he was pleased with the idea of one of the important ingredients, the quark model of hadrons. In 1966 Murray Gell-Mann, its proposer, visited Cambridge in order to spend a few months as an Overseas Fellow at Churchill College. On his arrival he went to pay his respects to Dirac and returned reporting that Dirac was delighted to hear of the role his relativistic wave equation was likely to play in view of the fact that electrons and quarks have the same spin. In retrospect it is clear that the introduction of the quark model marked the turning point back towards the Maxwell-Dirac theory in the history of the development of the standard model. Yet neither it, nor the unified electro-weak gauge theory of a few years later, were met with automatic acceptance by the sceptical particle physics community. Indeed Werner Heisenberg, for example, was quite antagonistic to the quark model, presumably because of his own rival theory, the non-linear spinor theory. I remember being taken aback when he publically reprimanded me in June 1971 for trying to justify string theory as a relativistic version of the quark model. “The quark model is not physics” were his words.

Of course there is a lot of devil in the detail covered by the word “elaboration” used above with reference to the Maxwell-Dirac system of equations and I shall say something about it. But the fact remains that physics had turned full circle. The reason was that, underlying all the complexity of the particle accelerator data, is a remarkably simple underlying principle that had been known in its embryonic form since its enunciation in 1929. Despite this, it had initially met with enormous scepticism when proposed in the wider context of the postwar data. Of course, I am talking about “the gauge principle” a notion that exploits the idea of locality that I have already promised to explain.

Before doing this I want to say how the gauge principle applied to the standard model brings into play something new compared to the earlier version. This is a branch of pure mathematics that is tailor-made to describe all the details neatly and succinctly and is known as the theory of compact Lie groups and their representations. The particular Lie groups occurring in the standard model are straightforward matrix groups called $U(2)$ and $SU(3)$ referring to the electroweak and strong forces respectively. These groups are formed
out of unitary matrices with two and three rows and columns and hence are continuous and have dimensions $2 \times 2 = 3 + 1$ and $3 \times 3 - 1 = 8$ respectively, equalling the number of gauge particles already mentioned when the photon is included. The quarks and leptons making up what was called the electron class in the explanation above likewise transform according to some natural representation of low dimension.

Thus all the data gathered at such expense and effort at particle accelerators around the world can be encapsulated by the gauge principle augmented by a few group theoretical rules. That is why the standard model is such an achievement. A plethora of *ad hoc* theories, four-fermi theory, Yukawa theory, isovector pion theory, flavour symmetry, V-A theory and so on, are all subsumed in, or superseded by one all encompassing model based on an aesthetic principle. The power of this revolution is that the evidence is provided by data produced by sceptics, namely the “hard-nosed” experimental physicists who have no truck with “airy-fairy” ideas such as the gauge principle.

It is important to realise that the principles of quantum mechanics and relativity remain intact and indeed are elevated to a larger range of validity whilst a new principle is included. The challenge emerges of finding an underlying explanation of the new picture with its rules and in developing the underlying principles further. Maybe even gravity can be incorporated. This pursuit is roughly known as “beyond the standard model” and it sets the stage for more aesthetic considerations and a renaissance of so many of Dirac’s pioneering ideas.

In order to introduce the work in which Dirac, with possibly his greatest stroke of genius, saw how to exploit the gauge principle to disclose aspects of quantum electrodynamics that still tantalise seventy years later, I shall recall the beginning of the story of quantum mechanics. It started with the radical proposal of Planck, clarified by Einstein, that the energy, $E$, of electromagnetic radiation of a given discrete angular frequency, $\omega$, occurred in quanta:

$$E = n\hbar\omega, \quad n = 0, 1, 2, 3, 4 \ldots$$

where $\hbar$ is Planck’s constant. It took almost thirty years of research to formulate principles of quantum mechanics sufficiently well that this result could be derived by applying these principles to Maxwell’s equations which were known to describe the radiation. Dirac was one of those who helped achieve this with his theory of second quantisation [Dirac 1927].

An obvious follow-up strategy was to survey natural phenomena and try to identify similar patterns of integer quantisation. One striking example was provided by the values of the electric charges, $q$, of the particles known to make up atoms. They were observed to follow an analogous rule:

$$q = nq_0, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

This rule explains why an atom in its natural un-ionised state is electrically neutral, a fact that is important for the stability of massive objects such as planets. But it is difficult to explain this result in terms of the principles of quantum mechanics alone since it is difficult to imagine a classical system in which electric charge is a dynamical variable in the way that the energy of radiation, in Planck’s relation above, is. So what could explain this pattern, whose validity survives the discovery of all the new post World War II particles, and is, in
fact, still the most striking feature of particle physics? Dirac found an answer in 1931 and it brought into play not one but two new ideas outside the framework of quantum mechanics, one of which was the gauge principle [Dirac 1931]. But it also brought a disappointment that continues to haunt theoretical particle physics today.

Now I come to explain what the gauge principle is, and something of the idea of locality. I have deferred this until I can no longer avoid introducing equations and formulae and I apologise in advance for this.

According to Schrödinger’s wave mechanical formulation of quantum theory, any electrically charged particle, such as an electron, possesses a wave function $\psi(x)$ that takes complex values. The argument $x$ denotes the space-time point at which the wave function is evaluated.

The wave function determines what is called the state of the electron, namely the totality of physical attributes that can be measured, but an awkward feature is that different (normalised) wave functions correspond to the same state if they differ by an overall phase factor, $e^{i\phi}$, that is independent of the space-time point, $x$. Thus, briefly, nothing physical is changed by the substitution:

$$\psi(x) \rightarrow e^{i\phi} \psi(x), \quad \phi \text{ constant.}$$

This feature is more disquieting when one remembers that, according to special relativity, no information signal can travel faster than the speed of light which is finite. So, if $x$ and $y$ are two distinct space-time points at the same time, there is no way that an observer at $x$ could know that the phase of the wave function had been altered at $y$. This consideration suggests that it would be more reasonable to require that the phase change in the substitution above could be allowed to vary over space-time:

$$\psi(x) \rightarrow e^{i\frac{q\chi(x)}{\hbar}} \psi(x).$$

So the phase change $q\chi(x)/\hbar$ now varies over space-time and so is said to be local. $q$ is the electric charge carried by the particle whose wave function is $\psi(x)$ and $\hbar$ is Planck’s constant, both inserted for convenience later on. The gauge principle states that all physical quantities should be unaltered by the substitution (2), which is the gauge transformation of the wave function. As we shall see, an awkward feature of wave mechanics has been converted into a new principle, in fact the most powerful principle of the latter half of the 20th century as far as particle theory is concerned.

Now the wave function has to satisfy some sort of evolution equation (such as the Schrödinger equation) involving its derivatives $\partial_\mu \psi = \frac{\partial \psi}{\partial x^\mu}$ with respect to the coordinates $x^\mu$ of the space-time point. According to the gauge principle, this equation has to have the property that if $\psi$ is a solution, then so is $e^{i\frac{q\chi(x)}{\hbar}} \psi$, whatever the function $\chi(x)$. This requires $\partial_\mu \psi$ to transform in the same way that $\psi$ does, as given by (2). This is impossible unless the notion of derivative is modified and replaced by what is called a “covariant derivative”:

$$D_\mu \psi = \partial_\mu \psi - \frac{iq}{\hbar} A_\mu \psi$$

with the understanding that in the evolution equation $\partial_\mu \psi$ always appears accompanied by $A_\mu$ in this combination. To understand the meaning of the newly introduced quantity $A_\mu$
it is necessary to see how it “gauge transforms”. Since we have found that the substitution (2) has to be accompanied by

\[ D_\mu \psi \rightarrow e^{i\chi(x)} D_\mu \psi(x) \]

a small calculation reveals that

\[ A_\mu \rightarrow A_\mu + \partial_\mu \chi, \tag{3} \]

the “gauge transformation” of \( A_\mu \). Notice that this equation, (3), has the property of being universal in the sense that it is independent of \( q \), the electric charge of the particle initially considered. In other words, the argument works for an array of particles with different electric charges, even if they do not satisfy the quantisation condition (1).

Notice also that the “curl” of \( A_\mu \) is unaltered by the gauge transformation (3):

\[ F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \rightarrow F_{\mu\nu}, \tag{4} \]

as \( \chi \) cancels out, by virtue of the irrelevance of the order of differentiation with respect to independent space-time variables. These final equations are very reminiscent of Einstein’s relativistic expression for the electromagnetic field tensor incorporating both the electric and magnetic fields. In fact it can be identified as precisely this. Then it follows that \( A_\mu \) is simply what is known as the gauge four-potential (composed of scalar and vector potentials). In fact the above gauge ambiguity (3) specified by \( \chi \) was well recognised in the context of Maxwell’s equations before the time of Planck’s proposal of quanta in 1900.

The identification of \( F_{\mu\nu} \) defined this way as the Einstein-Maxwell electromagnetic field strength tensor implies that the physical existence of this field is an inevitable consequence of the gauge principle, the idea that physics is unaffected by the local change of phase of wave functions given by (2). Furthermore the structure of the covariant derivative above determines how this field couples to any wave function and yields consequences perfectly consistent with all that was known previously.

All of this argument works equally well in curved space-time. The result endows the electromagnetic field with a new, geometrical significance that elevates it to become on a par with the gravitational field whose geometric meaning was found so famously by Einstein. The definitive version of this argument was found in 1929 by Hermann Weyl [1929], better known as a pure mathematician. More information about the genesis of the gauge principle, including the non-abelian version can be found in the recent book by O’Raifeartaigh [1997]. This also includes a translation into English of the paper by Hermann Weyl just mentioned. Dirac was very quick to exploit the ideas in a way that was completely novel, bringing into play ideas of topology for the first time in physics.

But first I want to explain how an extension of the gauge principle yields one of the basic ingredients of the standard model. Two successive gauge transformations (2) can be combined to yield a third at each point \( x \) of space-time

\[ e^{i\frac{\chi}{\hbar} (x)} e^{i\frac{\chi}{\hbar} (x)} = e^{i\frac{\chi}{\hbar} (x_1 + x_2(x))}. \]

This means that the transformations at any point form what is called a group, in this case the group of phase factors, usually denoted \( U(1) \) (unitary matrices with one row and
column). This particular group is abelian, that is the order of multiplication above does not matter, as is evident from the formula.

If one considers an array of particles, for example quarks and leptons, their wave functions can be arranged as a column vector with $N$ complex entries, say. A natural way to consider of extending (2) to this situation would be the following:

$$
\begin{pmatrix}
\psi_1(x) \\
\psi_2(x) \\
\psi_3(x) \\
\vdots \\
\psi_N(x)
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
D_{11} & D_{12} & D_{13} & \ldots & D_{1N} \\
D_{21} & D_{22} & D_{23} & \ldots & D_{2N} \\
D_{31} & D_{32} & D_{33} & \ldots & D_{3N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
D_{N1} & D_{N2} & D_{N3} & \ldots & D_{NN}
\end{pmatrix}
\begin{pmatrix}
\psi_1(x) \\
\psi_2(x) \\
\psi_3(x) \\
\vdots \\
\psi_N(x)
\end{pmatrix}
$$

that is, to premultiply the column vector by a square matrix with $N$ rows and columns and complex entries each dependent on the space-time point $x$. To preserve the total probability density this matrix should be unitary. Such substitutions again combine successively to form similar substitutions and so form a group at each point of space-time. As the order of multiplication now matters very much, the group is said to be non-abelian. The particular group formed in this way out of unitary matrices is called $U(N)$. If the determinants are constrained to equal unity a subgroup called $SU(N)$ results. These are examples of compact Lie groups and all other possibilities can be listed. But the examples just mentioned with $N$ taking the values 2 and 3 respectively suffice for the standard model.

One of the “devils in the detail” that I mentioned earlier is the non-abelian nature of these groups. The argument above leading to the gauge theory can be repeated and the field strength is found to be no longer linear in the gauge potential, as in (4), since additional quadratic terms occur. These mean that the equations of motion are now non-linear and so when, considered classically, possess unexpected solutions of a soliton character. The process of quantisation is made more difficult and this is one of the problems with which Dirac was preoccupied later in his career. The theory of constraints covered in his advanced lecture course was designed to aid the quantisation of theories with gauge symmetry and was taken up successfully by later workers.

The purpose of my digression into the standard model, which was probably the main achievement of the second half of the last century, was to make clear the importance within it of the gauge principle. In fact, this is pervasive in all modern thinking. Dirac used only the abelian, $U(1)$, version, first enunciated in 1929, by Hermann Weyl. Within two years Dirac had succeeded in exploiting it in the totally unexpected way that I shall now explain.

When the motion of an electrically charged particle is treated, ignoring quantum mechanics, it obeys Newton’s equation of motion in a form including the Lorenz force describing the effect of the electric and magnetic fields. This means that this equation, as well as the Maxwell equations, involves the field strength $F_{\mu\nu}$ directly without any independent appearance of the gauge potentials $A_\mu$. This situation does not change if a natural modification is made to include particles carrying a magnetic charge. Indeed there appears to be a symmetry with respect to the exchange of the roles of the electric and magnetic charges which is respected by the equations.

But the situation changes when quantum mechanics is introduced because of the way the gauge potentials explicitly enter the covariant derivative of the wave function of
the electrically charged particle, as explained above. Thus gauge potentials are needed to determine the evolution of the wave function and this makes it look as if quantum mechanics is inconsistent with the presence of particles with non-zero magnetic charge, \(g\). The reason is that the flux of magnetic field \(B\) through any surface, say a two-dimensional sphere, \(S_2\), surrounding a magnetic particle is

\[
\int_{S_2} B \cdot dS = g.
\]

On the sphere, \(S_2\), \(\nabla \cdot B\) vanishes as the magnetic charge is concentrated at one point, the position of the magnetically charged particle at its centre. This vanishing of \(\nabla \cdot B\) is necessary for the integrability of the equation \(B = \nabla \wedge A\), a special case of (4) with \(A\) denoting the vector potential, part of \(A_\mu\). But if this vector potential can be found all over the sphere then Stokes’ theorem implies that \(g\) vanishes. Dirac realised that this argument was wrong in a very subtle way. Instead of predicting \(g = 0\), he found that

\[
qg = 2\pi \hbar n, \quad n = 0, \pm 1, \pm 2 \pm 3 \ldots, \tag{5}
\]

for \(q\) being any electric charge and \(g\) any magnetic charge. So long as a particle with non-zero magnetic charge \(g\) exists in nature, this value can be divided out of equation (5) to yield equation (1), the quantisation condition for electric charge which very much agrees with observation. The attractiveness of this result evidently excited Dirac, but as he was aware, the price is the required existence of a magnetically charged particle. The repeated failure to observe isolated magnetic charge is what still haunts the study of this subject.

In the seventy years since Dirac derived his quantisation condition (5) there has been a minor industry in refining his argument to take account of notions developed later in pure mathematics. I shall present the treatment of Wu and Yang [1975].

It is the first part of the argument that brings topological ideas into play. Divide the sphere surrounding the magnetic charge into two hemispheres, north and south, which join on the equator. Unlike the sphere itself, each hemisphere has the property that it can be distorted continuously to its appropriate pole, north or south, without tearing. This means that it is valid to integrate the equation \(B = \nabla \wedge A\) to obtain the vector potential \(A\) separately in each hemisphere, \(A_{\text{NORTH}}\) and \(A_{\text{SOUTH}}\), respectively, even though this cannot be done for the sphere as a whole unless \(g\) vanishes, as we saw. It follows that the two vector potentials \(A_{\text{NORTH}}\) and \(A_{\text{SOUTH}}\) must differ on the equator despite the fact \(\nabla \wedge A_{\text{NORTH}}\) and \(\nabla \wedge A_{\text{SOUTH}}\) coincide there, both equalling the magnetic field which is well defined all over the sphere, since it is a physically observable quantity. This means that \(A_{\text{NORTH}}\) and \(A_{\text{SOUTH}}\) are related on the equator by a gauge transformation (3) which reads, in the present notation,

\[
A_{\text{NORTH}} = A_{\text{SOUTH}} + \nabla \chi.
\]

The magnetic charge equals the sum of the magnetic fluxes out of the two hemispheres and each of these can be evaluated using Stokes’ theorem for each hemisphere. Each contribution is an integral of the appropriate gauge potential, northern or southern, over
the boundary of the hemisphere which is, in both cases, the equator. In these integrals the equator has to have a definite sense and these must be relatively opposite since the sphere has no boundary, being a closed surface. Accordingly

\[
g = \int_{\text{NORTHERN HEMISPHERE}} B \cdot dS + \int_{\text{SOUTHERN HEMISPHERE}} B \cdot dS
\]

\[
= \int_{\text{EQUATOR}} (A^{\text{NORTH}} - A^{\text{SOUTH}}) \cdot dx = \int_{\text{EQUATOR}} \nabla \chi \cdot dx = \Delta \chi
\]

using the gauge transformation and an integration by parts. Thus the magnetic charge, \( g \), equals the jump, \( \Delta \chi \), in the value of the gauge function, \( \chi \), as the equator is encircled once.

The usefulness of this relation is enhanced when quantum mechanics is brought into play. Like the gauge potentials, the wave functions can be defined on each hemisphere separately but not all over the sphere at once. On the equator they must differ by a gauge transformation (2), and, in order to tally with the connection between the two gauge potentials above,

\[
\psi^{\text{NORTH}}(x) = e^{\frac{iq}{\hbar}} \chi(x) \psi^{\text{SOUTH}}(x)
\]

As each wave function is single valued in its own hemisphere, both are single-valued on the equator and it follows that the same also applies to the phase factor \( e^{\frac{iq}{\hbar}} \). Thus \( e^{\frac{iq}{\hbar}} \Delta \chi \) equals unity and this reduces to Dirac’s quantisation condition (5), when combined with the result above that \( \Delta \chi = g \).

The nature of the subtle extra ingredient supplied by Dirac, beyond quantum mechanics and the gauge principle involves ideas of what is now known as topology. In particular Dirac’s ideas bring into play global topology which is the antithesis of locality, involving the structure of spaces in the large rather than in the small. The mathematical subject of topology was in a rather primitive state in 1931 but has blossomed since then. With hindsight it can now be seen that in 1931 Dirac had anticipated ideas of de Rham cohomology, Hodge theory, fibre bundle theory and characteristic classes. I believe that Dirac was on friendly terms with his near contemporary, William Hodge, in the Cambridge of the 1930’s. Much later Hodge had a student, Atiyah, who was to discover remarkable connections between the Dirac equation and topology that have become an important part of modern quantum field theory. Indeed topological ideas now permeate that subject and even more modern string theory. This episode illustrates the relationship of Dirac’s thinking to pure mathematics. Although his mathematical thought was abstract and extremely effective, he seemed to dislike formalism per se, and it seems to me that the best description of his approach to pure mathematics was that it was very DIY, that is “do-it-yourself”.

I mentioned that the classical theory of particles carrying electric or magnetic charge possessed a formal symmetry with respect to the interchange of the electric and magnetic fields accompanied by a similar interchange of the two charges. The foregoing quantum argument seems to break the symmetry because of the preferred status of the electrically charged particles which are assigned wave functions while the magnetically charged particles are not, being treated as point objects. Despite this, the final result, Dirac’s quantisation condition, (5), does respect the symmetry with respect to interchange.
This prompts an interesting question as to whether a more complete theory, containing information concerning the structure and mass of these charged particles could also respect the symmetry. The advent of the standard model, or rather an extension known as a grand unified model, provides an arena in which an affirmative answer is possible.

Just as quantum electrodynamics is embedded as a subtheory in the standard model, so is the latter embedded in a grand unified model, namely a gauge theory with a simple Lie group as gauge group, \( G_{GUT} \), and hence fewer free parameters. Corresponding to this the relevant gauge groups are embedded in each other

\[ U(1)_Q \subset "U(2) \times SU(3)" \subset G_{GUT}. \]

\( Q \) is the generator of the electromagnetic gauge group, \( U(1) \), and hence automatically a generator of the Lie algebra of the grand unified Lie group. This fact has an important consequence: it can explain the quantisation of electric charge \( q \), apparently avoiding any need for magnetic charge. Indeed this was one of the original motives for considering such theories.

But it is necessary to ask how the direction of the electric charge is chosen out of the vector space of all directions in the Lie algebra of the grand unified group. The answer is that it is chosen by a special sort of Lorentz scalar field that fails to vanish in the vacuum, unlike the other fields. This is known as a “Higgs” field, and examples are already needed for other side-effects such as the provision of mass for the three weak gauge particles and the quarks whilst preserving full gauge invariance of the equations of motion.

When combined with the non-linearities of the generalised Maxwell equations, another effect of the Higgs field is the production of unexpected classical solutions, called solitons, that behave like stable, extended particles carrying magnetic charge, as ’t Hooft [1974] and Polyakov [1974] independently demonstrated. Thus a theory such as this, with electric charge quantisation inbuilt, automatically produces magnetically charged particle states.

Furthermore, Dirac’s quantisation condition \( (5) \) is seen to hold, arising in a way that has to do with the topology of the map provided by the Higgs field. Thus the survival of Dirac’s argument in a modern context demonstrates its pervasiveness.

In special cases at least, such as when \( G_{GUT} \equiv SO(3) \), the forementioned interchange symmetry between electric and magnetic charge does hold good, even when account is taken of expressions for the particle masses yielded by the Higgs effect [Montonen and Olive 1978]. As the single particle states satisfy the Dirac quantisation condition \( (5) \) with \( n = 2 \), this interchange reduces to

\[ q \rightarrow g = \frac{4\pi\hbar}{q}, \]

and is now manifestly quantum in nature. More remarkably there exist an infinite number of distinct, stable, single particle states carrying both electric and magnetic charge, and called “dyons” as a consequence. The theory exhibits a symmetry when these are permuted suitably. At the same time a new dimensionless parameter \( \theta \) enters in addition to the fine structure constant \( \alpha = \frac{q^2}{4\pi\hbar} \) and it turns out to be useful to unify the two by defining the following complex parameter:

\[ \tau = \frac{\theta}{2\pi} + \frac{i}{\alpha}. \]
It can be shown [Sen 1994] that the allowed permutations of dyonic particle states have the effect
\[ \tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \text{ integers,} \quad ad - bc = 1. \tag{8} \]

These fractional linear transformations form an infinite discrete group called the modular group, first studied by pure mathematicians in the 19th century. They realised that it had the important property of maintaining the positivity of the imaginary part of \( \tau \), physically the inverse of the fine structure constant according to expression (7), and so intrinsically positive.

To understand the transformations (8), first note that when \( \theta = 0 \) the transformation \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) yields the original interchange (6), whilst \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) simply increases \( \theta \) by \( 2\pi \), signifying that it is an angular variable as the notation suggests. Repetition of these two transformations yields all of the modular group.

According to the ideas of perturbation theory, it is possible to make good approximate calculations when the fine structure constant \( \alpha \) is small but not otherwise. So the behaviour of the theory when \( \alpha \) is large is considered inaccessible. But the transformation (6) interchanges large and small values of \( \alpha \) and so relates the unknown to the calculable. At first sight this is difficult to believe but a fair number of checks have confirmed the plausibility of this idea and it has become a matter of faith in much current research.

Another advantageous special condition on the spontaneously broken gauge theory is that it should display supersymmetry and this is easy to arrange. One consequence is that many of the divergences that troubled Dirac automatically cancel. Furthermore superstring theory contains such theories as a limiting case and, as a result, the ideas extend rather naturally to that framework. Making all that I have said more precise, watertight and wide-ranging is now a major priority in current research.

The influential, or even, prophetic nature of Dirac’s thinking is apparent in the above story. There are also examples in his later work that turned out to be likewise prophetic and/or influential. He was the first to advocate use of expressions in quantum electrodynamics that are akin to what are now called Wilson loops [Dirac 1955b]. In his bubble theory of the electron [Dirac 1962] he anticipated the Nambu-Goto action for the string and the later membrane action relevant to M-theory. His wave-front approach to quantum field theory [Dirac 1949] introduced the light-cone quantisation procedure applied so often subsequently. His analysis of the equal-time commutation relations of the energy-momentum tensor [Dirac 1955a] blossomed into the conformal field theory treatment of the critical exponents in the theory of second order phase transitions. During the 1950’s and 1960’s one major preoccupation of Dirac was the quantisation of constrained dynamical systems as a step towards the quantisation of systems with gauge symmetry such as Einstein’s theory of general relativity. Although the latter problem remains unsolved, his methods have proved valuable in the context of non-abelian gauge theories and hence in the standard model [Dirac 1950]. As I mentioned earlier he lectured on this subject in Cambridge and also wrote a book, newly republished [Dirac 1964]. The study of the strange representation of the anti-de-Sitter group [Dirac 1963] has burgeoned into the study of the metaplectic representation and something called Howe duality, quite possibly related to the electromagnetic duality explained above.
I have tried to explain briefly the pervasiveness of Dirac’s thought in modern theoretical physics by a few examples and, in particular, by reference to the development of the standard model and the era that followed. The latter example also illustrates the impact of the role played by Hermann Weyl, famous as a pure mathematician, but less recognised as a theoretical physicist.

Perspectives on the status of past scientific contributions are liable to change in the light of improvements in our own understanding following from general progress in the subject. I believe that the new perspective gained since the establishment of the standard model has enhanced Dirac’s reputation. Granted that he was one of the towering figures behind the development of quantum theory and relativistic quantum field theory, we can now see how his other ideas that were initially less appreciated are finally achieving similar degrees of appreciation and recognition for their ingenuity, correctness and relevance.

Finally I would like to thank Bruce Gordon for organising such a pleasant meeting.

References


H. Weyl, Z. Physik 56 (1929) 56.